

**Turbulent Flows**  
Stephen B. Pope  
*Cambridge University Press* (2000)

**Solution to Exercise H.4**

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*Date:* 04/24/03

With  $g'(\mathbf{v}; \mathbf{x}, t)$  being the fine-grained PDF  $\delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{v})$ , we obtain

$$\begin{aligned}
\nabla^2 g' &= \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} (\delta(\mathbf{u} - \mathbf{v})) = -\frac{\partial}{\partial x_k} \left( \frac{\partial g'}{\partial v_j} \frac{\partial u_j}{\partial x_k} \right) \\
&= -\frac{\partial g'}{\partial v_j} \frac{\partial^2 u_j}{\partial x_k \partial x_k} - \frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial v_j} \left( \frac{\partial g'}{\partial x_k} \right) \\
&= -\frac{\partial g'}{\partial v_j} \frac{\partial^2 u_j}{\partial x_k \partial x_k} + \frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial v_j} \left( \frac{\partial g'}{\partial v_i} \frac{\partial u_i}{\partial x_k} \right) \\
&= \frac{\partial^2 g'}{\partial v_i \partial v_j} \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} - \frac{\partial g'}{\partial v_j} \nabla^2 u_j.
\end{aligned} \tag{1}$$

With the above results, we obtain

$$\begin{aligned}
-\frac{\partial}{\partial v_i} \left( g \langle \nu \nabla^2 u_i | \mathbf{v} \rangle \right) &= -\left\langle \frac{\partial}{\partial v_i} \left( g' \nu \nabla^2 u_i \right) \right\rangle \\
&= -\nu \left\langle \frac{\partial g'}{\partial v_i} \nabla^2 u_i \right\rangle \\
&= \nu \left\langle \nabla^2 g' - \frac{\partial^2 g'}{\partial v_i \partial v_j} \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle \\
&= \nu \nabla^2 g - \nu \frac{\partial^2}{\partial v_i \partial v_j} \left( g \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \middle| \mathbf{v} \right\rangle \right) \\
&= \nu \nabla^2 g - \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left[ g \varepsilon_{ij}^c(\mathbf{v}) \right],
\end{aligned} \tag{2}$$

where  $\varepsilon_{ij}^c(\mathbf{v})$  is defined as

$$\varepsilon_{ij}^c(\mathbf{v}, \mathbf{x}, t) \equiv 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \middle| \mathbf{v} \right\rangle. \tag{3}$$

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