Turbulent Flows
Stephen B. Pope
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Solution to Exercise H.4

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With $g'(\mathbf{v};\mathbf{x},t)$ being the fine-grained PDF $\delta\left(\mathbf{u}\left(\mathbf{x},t\right)-\mathbf{v}\right)$, we obtain

$$\begin{aligned}
\nabla^2 g' &= \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\left(\delta(\mathbf{u}-\mathbf{v})\right) = -\frac{\partial}{\partial x_k}\left(\frac{\partial g'}{\partial v_j}\frac{\partial u_j}{\partial x_k}\right) \\
&= -\frac{\partial g'}{\partial v_j}\frac{\partial^2 u_j}{\partial x_k \partial x_k} - \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial v_j}\left(\frac{\partial g'}{\partial x_k}\right) \\
&= -\frac{\partial g'}{\partial v_j}\frac{\partial^2 u_j}{\partial x_k \partial x_k} + \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial v_j}\left(\frac{\partial g'}{\partial v_i}\frac{\partial u_i}{\partial x_k}\right) \\
&= \frac{\partial^2 g'}{\partial v_i \partial v_j}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k} - \frac{\partial g'}{\partial v_j}\nabla^2 u_j. \qquad (1)
\end{aligned}$$

With the above results, we obtain

$$\begin{aligned}
-\frac{\partial}{\partial v_i}\Big(g\langle \nu\nabla^2 u_i|\mathbf{v}\rangle\Big) &= -\langle \frac{\partial}{\partial v_i}\Big(g'\nu\nabla^2 u_i\Big)\rangle \\
&= -\nu\langle \frac{\partial g'}{\partial v_i}\nabla^2 u_i\rangle \\
&= \nu\left\langle \nabla^2 g' - \frac{\partial^2 g'}{\partial v_i \partial v_j}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right\rangle \\
&= \nu\nabla^2 g - \nu\frac{\partial^2}{\partial v_i \partial v_j}\left(g\left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\;\Big|\;\mathbf{v}\right\rangle\right) \\
&= \nu\nabla^2 g - \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\Big[g\varepsilon^c_{ij}(\mathbf{v})\Big], \qquad (2)
\end{aligned}$$

where $\varepsilon^c_{ij}(\mathbf{v})$ is defined as

$$\varepsilon^c_{ij}(\mathbf{v},\mathbf{x},t) \equiv 2\nu\left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\;\Big|\;\mathbf{v}\right\rangle. \qquad (3)$$