Turbulent Flows
Stephen B. Pope
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Solution to Exercise J.5

Prepared by: A.G. Lamorgese and S.B. Pope *Date:* 5/11/03

From Eq. (J.56), the JPDF of $\mathbf{U}(t)$, $f(\mathbf{V};t)$, satisfies:

$$\frac{\partial f}{\partial t} = \frac{1}{T}\frac{\partial (fV_i)}{\partial V_i} + \frac{\sigma^2}{T}\frac{\partial^2 f}{\partial V_i \partial V_i}. \tag{1}$$

With the deterministic initial condition, $f_1(\mathbf{V};t_1|\mathbf{V}_1,t_1) = \delta(\mathbf{V}-\mathbf{V}_1)$, the solution is a joint normal distribution, with mean vector and covariance matrix given by $\mathbf{V}_1 e^{-(t-t_1)/T}$ and $\delta_{ij}\sigma^2(1-e^{-2(t-t_1)/T})$, respectively.

To obtain the SDE for $q(t) = (U_i(t)U_i(t))^{1/2}$, let us consider

$$d(q^2) \quad = (U_i + dU_i)(U_i + dU_i) - U_iU_i = dU_idU_i + 2U_idU_i \tag{2}$$

$$= \tfrac{2\sigma^2}{T}\delta_{kl}dW_kdW_l - 2\tfrac{q^2}{T}dt + 2\left(\tfrac{2\sigma^2}{T}\right)^{1/2} U_idW_i + o(dt) \tag{3}$$

Now U_idW_i has zero mean and variance q^2dt, so it can expressed as $q(t)dW(t)$, where $dW(t)$ is a scalar-valued Wiener process independent of $dW_i(t)$. Also, $\langle dW_idW_i\rangle = 3dt$; thus Eq. (2) can be recast in the form:

$$dq(t) = \left(\frac{3\sigma^2}{q} - q\right)\frac{dt}{T} + \left(\frac{2\sigma^2}{T}\right)^{1/2} dW(t) \tag{4}$$

The stationary distribution of q follows from Eq. (J.21), a and b^2 being the drift and diffusion coefficients associated with Eq. (**??**).

Thus,

$$f(v) = c_1 \exp\left(3\ln\frac{v}{v_0} - \frac{v^2}{2\sigma^2}\right) = c_1' v^3 e^{-v^2/(2\sigma^2)} \tag{5}$$

with $c_1' = 1/(2\sigma^4)$.