

Turbulent Flows
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Solution to Exercise J.5

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From Eq. (J.56), the JPDF of $\mathbf{U}(t)$, $f(\mathbf{V}; t)$, satisfies:

$$\frac{\partial f}{\partial t} = \frac{1}{T} \frac{\partial (fV_i)}{\partial V_i} + \frac{\sigma^2}{T} \frac{\partial^2 f}{\partial V_i \partial V_i}. \quad (1)$$

With the deterministic initial condition, $f_1(\mathbf{V}; t_1 | \mathbf{V}_1, t_1) = \delta(\mathbf{V} - \mathbf{V}_1)$, the solution is a joint normal distribution, with mean vector and covariance matrix given by $\mathbf{V}_1 e^{-(t-t_1)/T}$ and $\delta_{ij} \sigma^2 (1 - e^{-2(t-t_1)/T})$, respectively.

To obtain the SDE for $q(t) = (U_i(t)U_i(t))^{1/2}$, let us consider

$$d(q^2) = (U_i + dU_i)(U_i + dU_i) - U_i U_i = dU_i dU_i + 2U_i dU_i \quad (2)$$

$$= \frac{2\sigma^2}{T} \delta_{kl} dW_k dW_l - 2\frac{q^2}{T} dt + 2 \left(\frac{2\sigma^2}{T} \right)^{1/2} U_i dW_i + o(dt) \quad (3)$$

Now $U_i dW_i$ has zero mean and variance $q^2 dt$, so it can be expressed as $q(t) dW(t)$, where $dW(t)$ is a scalar-valued Wiener process independent of $dW_i(t)$. Also, $\langle dW_i dW_i \rangle = 3dt$; thus Eq. (2) can be recast in the form:

$$dq(t) = \left(\frac{3\sigma^2}{q} - q \right) \frac{dt}{T} + \left(\frac{2\sigma^2}{T} \right)^{1/2} dW(t) \quad (4)$$

The stationary distribution of q follows from Eq. (J.21), a and b^2 being the drift and diffusion coefficients associated with Eq. (4).

Thus,

$$f(v) = c_1 \exp \left(3 \ln \frac{v}{v_0} - \frac{v^2}{2\sigma^2} \right) = c'_1 v^3 e^{-v^2/(2\sigma^2)} \quad (5)$$

with $c'_1 = 1/(2\sigma^4)$.

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