Turbulent Flows
Stephen B. Pope
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Solution to Exercise 10.13

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Eq.(10.53):

$$\frac{\overline{D}\varepsilon}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \tag{1}$$

Substitute $\omega = \varepsilon/k$ into Eq.(1):

$$\frac{\overline{D}(\omega k)}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + C_{\varepsilon 1} \mathcal{P}\omega - C_{\varepsilon 2} \omega^2 k \tag{2}$$

$$k\frac{\overline{D}\omega}{\overline{D}t} + \omega \frac{\overline{D}k}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + C_{\varepsilon 1} \mathcal{P}\omega - C_{\varepsilon 2} \omega^2 k \tag{3}$$

Eq.(10.41):

$$\frac{\overline{D}k}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \mathcal{P} - \varepsilon \tag{4}$$

Substitute Eq.(4) into Eq.(3) using $\omega = \varepsilon/k$:

$$k\frac{\overline{D}\omega}{\overline{D}t} + \omega \left(\nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \mathcal{P} - \varepsilon \right) = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + C_{\varepsilon 1} \mathcal{P}\omega - C_{\varepsilon 2} \omega^2 k \tag{5}$$

Divide by k and rearrange:

$$\frac{\overline{D}\omega}{\overline{D}t} = -\frac{\omega}{k} \nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \frac{1}{k} \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + (C_{\varepsilon 1} - 1) \frac{\mathcal{P}\omega}{k} - (C_{\varepsilon 2} - 1)\omega^2 \tag{6}$$

Eq.(10.47)

$$\nu_t = C_\mu k^2/\varepsilon \tag{7}$$

Consider first term on the RHS of Eq.(6) $-\frac{\omega}{k}\nabla\cdot\left(\frac{\nu_t}{\sigma_k}\nabla k\right)$ and substitute Eq.(7) and $\omega=\varepsilon/k$:

$$-\frac{\omega}{k}\nabla\cdot\left(\frac{\nu_t}{\sigma_k}\nabla k\right)=-\frac{\omega}{k}\nabla\cdot\left(\frac{C_\mu k^2}{\varepsilon\sigma_k}\nabla k\right)=-\frac{\omega}{k}\nabla\cdot\left(\frac{C_\mu k}{\sigma_k\omega}\nabla k\right)= \\ -\frac{\omega}{k}\frac{C_\mu}{\sigma_k}\left(\frac{\nabla k\nabla k}{\omega}+\frac{k\nabla^2 k}{\omega}-\frac{k\nabla k\nabla\omega}{\omega^2}\right)=-\frac{C_\mu}{\sigma_k}\left(\frac{\nabla k\nabla k}{\omega}+\nabla^2 k- \right. \\ \left. -\frac{k\nabla k\nabla}{\omega}\right) \tag{8}$$

Similarly for second term on the RHS $\frac{1}{k}\nabla\cdot\left(\frac{\nu_t}{\sigma_\varepsilon}\nabla(\omega k)\right)$:

$$\frac{1}{k}\nabla\cdot\left(\frac{\nu_t}{\sigma_\varepsilon}\nabla(\omega k)\right)=\frac{C_\mu}{\sigma_\varepsilon}\frac{1}{k}\nabla\cdot\left(\frac{k}{\omega}\nabla(\omega k)\right)=\frac{C_\mu}{\sigma_\varepsilon}\frac{1}{k}\nabla\cdot\left(\frac{k}{\omega}(\omega\nabla k+ \right. \\ \left. k\nabla\omega)\right)=\frac{C_\mu}{\sigma_\varepsilon}\frac{1}{k}\nabla\cdot\left(k\frac{k\nabla\omega}{\omega}+k\nabla k\right)=\frac{1}{k}\frac{C_\mu}{\sigma_\varepsilon}\left(\nabla k\frac{k\nabla\omega}{\omega}+k\nabla\left(\frac{k\nabla\omega}{\omega}\right)+ \right. \\ \left. k\nabla^2 k+\nabla k\cdot\nabla k\right)=\frac{C_\mu}{\sigma_\varepsilon}\frac{\nabla k\nabla\omega}{\omega}+\frac{1}{\sigma_\varepsilon}\nabla\left(\frac{C_\mu k^2}{\varepsilon}\nabla\omega\right)+\frac{C_\mu}{\sigma_\varepsilon}\nabla^2 k+\frac{C_\mu}{\sigma_\varepsilon}\frac{\nabla k\cdot\nabla k}{k} \tag{9}$$

By substituting Eqs.(8-9) into Eq.6 and rearranging obtain:

$$\frac{\overline{D}\omega}{\overline{D}t}=\nabla\cdot\left(\frac{\nu_t}{\sigma_\varepsilon}\nabla\omega\right)+(C_{\varepsilon 1}-1)\frac{\mathcal{P}\omega}{k}-(C_{\varepsilon 2}-1)\omega^2+ \\ C_\mu\left(\frac{1}{\sigma_\varepsilon}+\frac{1}{\sigma_k}\right)\frac{1}{\omega}\nabla\omega\cdot\nabla k+C_\mu\left(\frac{1}{\sigma_\varepsilon}-\frac{1}{\sigma_k}\right)\left(\nabla^2 k+\frac{1}{k}\nabla k\cdot\nabla k\right) \tag{10}$$