

Turbulent Flows
 Stephen B. Pope
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Solution to Exercise 10.13

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Eq.(10.53):

$$\frac{\overline{D}\varepsilon}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \quad (1)$$

Substitute $\omega = \varepsilon/k$ into Eq.(1):

$$\frac{\overline{D}(\omega k)}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + C_{\varepsilon 1} \mathcal{P}\omega - C_{\varepsilon 2} \omega^2 k \quad (2)$$

$$k \frac{\overline{D}\omega}{\overline{D}t} + \omega \frac{\overline{D}k}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + C_{\varepsilon 1} \mathcal{P}\omega - C_{\varepsilon 2} \omega^2 k \quad (3)$$

Eq.(10.41):

$$\frac{\overline{D}k}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \mathcal{P} - \varepsilon \quad (4)$$

Substitute Eq.(4) into Eq.(3) using $\omega = \varepsilon/k$:

$$k \frac{\overline{D}\omega}{\overline{D}t} + \omega \left(\nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \mathcal{P} - \varepsilon \right) = \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + C_{\varepsilon 1} \mathcal{P}\omega - C_{\varepsilon 2} \omega^2 k \quad (5)$$

Divide by k and rearrange:

$$\frac{\overline{D}\omega}{\overline{D}t} = -\frac{\omega}{k} \nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \frac{1}{k} \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla(\omega k) \right) + (C_{\varepsilon 1} - 1) \frac{\mathcal{P}\omega}{k} - (C_{\varepsilon 2} - 1) \omega^2 \quad (6)$$

Eq.(10.47)

$$\nu_t = C_\mu k^2 / \varepsilon \quad (7)$$

Consider first term on the RHS of Eq.(6) $-\frac{\omega}{k}\nabla \cdot \left(\frac{\nu_t}{\sigma_k}\nabla k\right)$ and substitute Eq.(7) and $\omega = \varepsilon/k$:

$$\begin{aligned} -\frac{\omega}{k}\nabla \cdot \left(\frac{\nu_t}{\sigma_k}\nabla k\right) &= -\frac{\omega}{k}\nabla \cdot \left(\frac{C_\mu k^2}{\varepsilon\sigma_k}\nabla k\right) = -\frac{\omega}{k}\nabla \cdot \left(\frac{C_\mu k}{\sigma_k\omega}\nabla k\right) = \\ &-\frac{\omega}{k}\frac{C_\mu}{\sigma_k}\left(\frac{\nabla k\nabla k}{\omega} + \frac{k\nabla^2 k}{\omega} - \frac{k\nabla k\nabla\omega}{\omega^2}\right) = -\frac{C_\mu}{\sigma_k}\left(\frac{\nabla k\nabla k}{\omega} + \nabla^2 k - \right. \quad (8) \\ &\quad \left. -\frac{k\nabla k\nabla}{\omega}\right) \end{aligned}$$

Similarly for second term on the RHS $\frac{1}{k}\nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon}\nabla(\omega k)\right)$:

$$\begin{aligned} \frac{1}{k}\nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon}\nabla(\omega k)\right) &= \frac{C_\mu}{\sigma_\varepsilon}\frac{1}{k}\nabla \cdot \left(\frac{k}{\omega}\nabla(\omega k)\right) = \frac{C_\mu}{\sigma_\varepsilon}\frac{1}{k}\nabla \cdot \left(\frac{k}{\omega}(\omega\nabla k + \right. \\ &k\nabla\omega)\left.)\right) = \frac{C_\mu}{\sigma_\varepsilon}\frac{1}{k}\nabla \cdot \left(k\frac{k\nabla\omega}{\omega} + k\nabla k\right) = \frac{1}{k}\frac{C_\mu}{\sigma_\varepsilon}\left(\nabla k\frac{k\nabla\omega}{\omega} + k\nabla\left(\frac{k\nabla\omega}{\omega}\right) + \right. \\ &k\nabla^2 k + \nabla k \cdot \nabla k\left.)\right) = \frac{C_\mu}{\sigma_\varepsilon}\frac{\nabla k\nabla\omega}{\omega} + \frac{1}{\sigma_\varepsilon}\nabla\left(\frac{C_\mu k^2}{\varepsilon}\nabla\omega\right) + \frac{C_\mu}{\sigma_\varepsilon}\nabla^2 k + \frac{C_\mu}{\sigma_\varepsilon}\frac{\nabla k \cdot \nabla k}{k} \quad (9) \end{aligned}$$

By substituting Eqs.(8-9) into Eq.6 and rearranging obtain:

$$\begin{aligned} \frac{\overline{D}\omega}{\overline{D}t} &= \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon}\nabla\omega\right) + (C_{\varepsilon 1} - 1)\frac{\mathcal{P}\omega}{k} - (C_{\varepsilon 2} - 1)\omega^2 + \\ &C_\mu\left(\frac{1}{\sigma_\varepsilon} + \frac{1}{\sigma_k}\right)\frac{1}{\omega}\nabla\omega \cdot \nabla k + C_\mu\left(\frac{1}{\sigma_\varepsilon} - \frac{1}{\sigma_k}\right)\left(\nabla^2 k + \frac{1}{k}\nabla k \cdot \nabla k\right) \quad (10) \end{aligned}$$

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