Turbulent Flows
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Solution to Exercise 11.13

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a) According to Eqs. (11.80), (11.81) and (11.82), we get

$$\frac{d\hat{\kappa}_\ell}{dt} = -\hat{\kappa}_j \frac{\partial \langle U_j \rangle}{\partial x_\ell}, \tag{1}$$

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(t) e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}, \tag{2}$$

and

$$p^{(r)}(\mathbf{x}, t) = \hat{p}(t) e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}. \tag{3}$$

According to Eq. (11.69), we get

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2\frac{\partial \langle U_i \rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \tag{4}$$

Substituting Eqs 2 and 3 into Eq. 4, we get

$$\nabla^2 \left(\hat{p}(t) e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\right) = -2\rho \frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial \left(\hat{u}_j(t) e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\right)}{\partial x_i}, \tag{5}$$

i.e.,

$$\hat{p}(t)\hat{\kappa}_\ell\hat{\kappa}_\ell e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}} = 2i\rho\frac{\partial \langle U_i \rangle}{\partial x_j}\hat{u}_j(t) e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\hat{\kappa}_i. \tag{6}$$

From Eq. 6, we get

$$\hat{\kappa}^2\hat{p} = 2i\rho\hat{\kappa}_i\hat{u}_j\frac{\partial \langle U_i \rangle}{\partial x_j}, \tag{7}$$

i.e.,

$$\hat{\kappa}^2\hat{p} = i2\rho\hat{\kappa}_\ell\hat{u}_k\frac{\partial \langle U_\ell \rangle}{\partial x_k}. \tag{8}$$

b) With Eq. 8, we have

$$\begin{aligned}
-\frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial\left(\hat{p}(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\right)}{\partial x_j} \\
&= -i\frac{\hat{p}(t)}{\rho}e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\hat{\kappa}_j \\
&= -i\frac{1}{\rho}e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\hat{\kappa}_j\left(\frac{i2\rho\hat{\kappa}_\ell\hat{u}_k}{\hat{\kappa}^2}\right)\frac{\partial\langle U_\ell\rangle}{\partial x_k} \\
&= 2\frac{\hat{\kappa}_j\hat{\kappa}_\ell}{\hat{\kappa}^2}u_k\frac{\partial\langle U_\ell\rangle}{\partial x_k}. \qquad (9)
\end{aligned}$$

c) With Eq. 2 and Eq.(11.87), we get

$$\begin{aligned}
\frac{\bar{D}u_j}{\bar{D}t} &= \frac{\partial}{\partial t}\left(\hat{u}_j(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\right) + \langle U_\ell\rangle\frac{\partial}{\partial x_\ell}\left(\hat{u}_j(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\right) \\
&= e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\frac{\partial\hat{u}_j(t)}{\partial t} + i\hat{u}_j(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}x_\ell\frac{d\kappa_\ell}{dt} + i\langle U_\ell\rangle\hat{u}_j(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\kappa_\ell \\
&= e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\frac{d\hat{u}_j(t)}{\partial t} - i\hat{u}_j(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}x_\ell\hat{\kappa}_m\frac{\partial\langle U_m\rangle}{\partial x_\ell} + i\frac{\partial\langle U_\ell\rangle}{\partial x_m}x_m\hat{u}_j(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\kappa_\ell \\
&= e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\frac{d\hat{u}_j}{dt}. \qquad (10)
\end{aligned}$$

d) According to the rapid-distortion equation, we get

$$\frac{\bar{D}u_j}{\bar{D}t} = -u_i\frac{\partial\langle U_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_j}. \qquad (11)$$

Substituting Eqs. 10, 9 and 2 into Eq. 11, we get

$$e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\frac{d\hat{u}_j}{dt} = -\hat{u}_i e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\frac{\partial\langle U_j\rangle}{\partial x_i} + 2\frac{\hat{\kappa}_j\hat{\kappa}_\ell}{\hat{\kappa}^2}u_k\frac{\partial\langle U_\ell\rangle}{\partial x_k}, \qquad (12)$$

i.e.,

$$e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\frac{d\hat{u}_j}{dt} = -\hat{u}_i e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\frac{\partial\langle U_j\rangle}{\partial x_i} + 2\frac{\hat{\kappa}_j\hat{\kappa}_\ell}{\hat{\kappa}^2}\hat{u}_k e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\frac{\partial\langle U_\ell\rangle}{\partial x_k}. \qquad (13)$$

From Eq. 13, we finally get

$$\frac{d\hat{u}_j}{dt} = -\hat{u}_k\frac{\partial\langle U_\ell\rangle}{\partial k_k}\left(\delta_{j\ell} - 2\frac{\hat{\kappa}_j\hat{\kappa}_\ell}{\hat{\kappa}^2}\right). \qquad (14)$$

So the rapid-distortion equation is satisfied if $\hat{\mathbf{u}}(t)$ evolves by Eq. 14.