

Turbulent Flows
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Solution to Exercise 11.13

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a) According to Eqs. (11.80), (11.81) and (11.82), we get

$$\frac{d\hat{\kappa}_\ell}{dt} = -\hat{\kappa}_j \frac{\partial \langle U_j \rangle}{\partial x_\ell}, \quad (1)$$

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(t) e^{i\hat{\mathbf{\kappa}}(t) \cdot \mathbf{x}}, \quad (2)$$

and

$$p^{(r)}(\mathbf{x}, t) = \hat{p}(t) e^{i\hat{\mathbf{\kappa}}(t) \cdot \mathbf{x}}. \quad (3)$$

According to Eq. (11.69), we get

$$\frac{1}{\rho} \nabla^2 p^{(r)} = -2 \frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial u_j}{\partial x_i}. \quad (4)$$

Substituting Eqs 2 and 3 into Eq. 4, we get

$$\nabla^2 \left(\hat{p}(t) e^{i\hat{\mathbf{\kappa}}(t) \cdot \mathbf{x}} \right) = -2\rho \frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial \left(\hat{u}_j(t) e^{i\hat{\mathbf{\kappa}}(t) \cdot \mathbf{x}} \right)}{\partial x_i}, \quad (5)$$

i.e.,

$$\hat{p}(t) \hat{\kappa}_\ell \hat{\kappa}_\ell e^{i\hat{\mathbf{\kappa}}(t) \cdot \mathbf{x}} = 2i\rho \frac{\partial \langle U_i \rangle}{\partial x_j} \hat{u}_j(t) e^{i\hat{\mathbf{\kappa}}(t) \cdot \mathbf{x}} \hat{\kappa}_i. \quad (6)$$

From Eq. 6, we get

$$\hat{\kappa}^2 \hat{p} = 2i\rho \hat{\kappa}_i \hat{u}_j \frac{\partial \langle U_i \rangle}{\partial x_j}, \quad (7)$$

i.e.,

$$\hat{\kappa}^2 \hat{p} = i2\rho \hat{\kappa}_\ell \hat{u}_k \frac{\partial \langle U_\ell \rangle}{\partial x_k}. \quad (8)$$

b) With Eq. 8, we have

$$\begin{aligned}
-\frac{1}{\rho} \frac{\partial p^{(r)}}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial \left(\hat{p}(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \right)}{\partial x_j} \\
&= -i \frac{\hat{p}(t)}{\rho} e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \hat{k}_j \\
&= -i \frac{1}{\rho} e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \hat{k}_j \left(\frac{i2\rho \hat{k}_\ell \hat{u}_k}{\hat{k}^2} \right) \frac{\partial \langle U_\ell \rangle}{\partial x_k} \\
&= 2 \frac{\hat{k}_j \hat{k}_\ell}{\hat{k}^2} u_k \frac{\partial \langle U_\ell \rangle}{\partial x_k}.
\end{aligned} \tag{9}$$

c) With Eq. 2 and Eq.(11.87), we get

$$\begin{aligned}
\frac{\bar{D}u_j}{Dt} &= \frac{\partial}{\partial t} \left(\hat{u}_j(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \right) + \langle U_\ell \rangle \frac{\partial}{\partial x_\ell} \left(\hat{u}_j(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \right) \\
&= e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{\partial \hat{u}_j(t)}{\partial t} + i \hat{u}_j(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} x_\ell \frac{d\kappa_\ell}{dt} + i \langle U_\ell \rangle \hat{u}_j(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \kappa_\ell \\
&= e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{d\hat{u}_j(t)}{dt} - i \hat{u}_j(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} x_\ell \hat{k}_m \frac{\partial \langle U_m \rangle}{\partial x_\ell} + i \frac{\partial \langle U_\ell \rangle}{\partial x_m} x_m \hat{u}_j(t) e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \kappa_\ell \\
&= e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{d\hat{u}_j}{dt}.
\end{aligned} \tag{10}$$

d) According to the rapid-distortion equation, we get

$$\frac{\bar{D}u_j}{Dt} = -u_i \frac{\partial \langle U_j \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial p^{(r)}}{\partial x_j}. \tag{11}$$

Substituting Eqs. 10, 9 and 2 into Eq. 11, we get

$$e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{d\hat{u}_j}{dt} = -\hat{u}_i e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{\partial \langle U_j \rangle}{\partial x_i} + 2 \frac{\hat{k}_j \hat{k}_\ell}{\hat{k}^2} u_k \frac{\partial \langle U_\ell \rangle}{\partial x_k}, \tag{12}$$

i.e.,

$$e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{d\hat{u}_j}{dt} = -\hat{u}_i e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{\partial \langle U_j \rangle}{\partial x_i} + 2 \frac{\hat{k}_j \hat{k}_\ell}{\hat{k}^2} \hat{u}_k e^{i\hat{\mathbf{k}}(t) \cdot \mathbf{x}} \frac{\partial \langle U_\ell \rangle}{\partial x_k}. \tag{13}$$

From Eq. 13, we finally get

$$\frac{d\hat{u}_j}{dt} = -\hat{u}_k \frac{\partial \langle U_\ell \rangle}{\partial k_k} \left(\delta_{j\ell} - 2 \frac{\hat{k}_j \hat{k}_\ell}{\hat{k}^2} \right). \tag{14}$$

So the rapid-distortion equation is satisfied if $\hat{\mathbf{u}}(t)$ evolves by Eq. 14.

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