Turbulent Flows
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Solution to Exercise 11.14

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From Eq. (11.81), we get

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}. \tag{1}$$

Substituting Eq. 1 into the continuity equation, we get

$$\begin{aligned}\nabla.\mathbf{u} = 0 &= \frac{\partial}{\partial x_i}\left(\hat{u}_i(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\right) \\ &= i\hat{u}_i(t)e^{i\hat{\boldsymbol{\kappa}}(t)\cdot\mathbf{x}}\hat{\kappa}_i(t). \end{aligned}\tag{2}$$

From Eq. 2, we get

$$\hat{\boldsymbol{\kappa}}(t)\cdot\hat{\mathbf{u}}(t) = 0. \tag{3}$$

So the continuity equation requires that $\hat{\boldsymbol{\kappa}}(t)$ and $\hat{\mathbf{u}}(t)$ be orthogonal.
According to Eqs. (11.80) and (11.83), we get

$$\frac{d\hat{\kappa}_m}{dt} = -\hat{\kappa}_\ell\frac{\partial\langle U_\ell\rangle}{\partial x_m}, \tag{4}$$

and

$$\frac{d\hat{u}_j}{dt} = -\hat{u}_k\frac{\partial\langle U_\ell\rangle}{\partial k_k}\left(\delta_{j\ell} - 2\frac{\hat{\kappa}_j\hat{\kappa}_\ell}{\hat{\kappa}^2}\right). \tag{5}$$

Multiplying Eq. 4 with $\hat{u}_j$ and Eq. 5 with $\hat{\kappa}_m$, we get

$$\hat{u}_j\frac{d\hat{\kappa}_m}{dt} = -\hat{u}_j\hat{\kappa}_\ell\frac{\partial\langle U_\ell\rangle}{\partial x_m}, \tag{6}$$

and

$$\hat{\kappa}_m\frac{d\hat{u}_j}{dt} = -\hat{\kappa}_m\hat{u}_k\frac{\partial\langle U_\ell\rangle}{\partial x_k}\left(\delta_{j\ell} - 2\frac{\hat{\kappa}_j\hat{\kappa}_\ell}{\hat{\kappa}^2}\right). \tag{7}$$

Adding Eq. 6 to Eq. 7, we get

$$\frac{d\left(\hat{\kappa}_m \hat{u}_j\right)}{dt} = -\hat{u}_j \hat{\kappa}_\ell \frac{\partial \langle U_\ell \rangle}{\partial x_m} - \hat{\kappa}_m \hat{u}_k \frac{\partial \langle U_\ell \rangle}{\partial x_k} \left(\delta_{j\ell} - 2 \frac{\hat{\kappa}_j \hat{\kappa}_\ell}{\hat{\kappa}^2} \right) \tag{8}$$

Multiplying Eq. 8 with δ_{mj}, we get

$$\begin{aligned} \frac{d\left(\hat{\kappa}_j \hat{u}_j\right)}{dt} &= -\hat{u}_j \hat{\kappa}_\ell \frac{\partial \langle U_\ell \rangle}{\partial x_j} - \hat{\kappa}_j \hat{u}_k \frac{\partial \langle U_j \rangle}{\partial x_k} + 2 \hat{u}_k \hat{\kappa}_j \frac{\partial \langle U_\ell \rangle}{\partial x_k} \frac{\hat{\kappa}_j \hat{\kappa}_\ell}{\hat{\kappa}^2} \\ &= 0. \end{aligned} \tag{9}$$

So given that this condition is satisfied initially (i.e., $\hat{\boldsymbol{\kappa}}(0) \cdot \hat{\mathbf{u}}(0) = 0$), the evolution equations for $\hat{\boldsymbol{\kappa}}(t)$ and $\hat{\mathbf{u}}(t)$ maintain this orthogonality.