

Turbulent Flows
Stephen B. Pope
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Solution to Exercise 11.18

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For isotropic turbulence, the most general possible (isotropic) expression for M_{ijkl} is

$$M_{ijkl} = k(\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}), \quad (1)$$

where α , β and γ are constants.

From Exercise 11.17, we know that M_{ijkl} satisfies the following conditions:

$$M_{ijkl} = M_{jikl}, \quad M_{ijkl} = M_{ijlk}, \quad M_{ijjl} = 0, \quad M_{ijkk} = \langle u_i u_j \rangle. \quad (2)$$

With $M_{ijkl} = M_{jikl}$, we get

$$k(\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}) = k(\alpha\delta_{ji}\delta_{kl} + \beta\delta_{jk}\delta_{il} + \gamma\delta_{jl}\delta_{ik}). \quad (3)$$

So from Eq. 3, we get

$$\beta = \gamma. \quad (4)$$

With $M_{ijjl} = 0$, we get

$$k(\alpha\delta_{ij}\delta_{jl} + \beta\delta_{ij}\delta_{jl} + 3\gamma\delta_{il}) = 0. \quad (5)$$

So from Eq. 5, we get

$$\alpha + \beta + 3\gamma = 0. \quad (6)$$

With $M_{ijkk} = \langle u_i u_j \rangle$, we get

$$\begin{aligned} M_{ijkk} &= k(\alpha\delta_{ij}\delta_{kk} + \beta\delta_{ik}\delta_{jk} + \gamma\delta_{ik}\delta_{jk}) \\ &= \langle u_i u_j \rangle = \frac{2}{3}k\delta_{ij}, \end{aligned} \quad (7)$$

where the last step follows from the isotropic turbulence.

So from Eq. 7, we get

$$3\alpha + \beta + \gamma = \frac{2}{3}. \quad (8)$$

With Eqs. 4, 6 and 8, we get

$$\alpha = \frac{4}{15}, \quad \beta = \gamma = -\frac{1}{15}. \quad (9)$$

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