Turbulent Flows
Stephen B. Pope
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Solution to Exercise 11.18

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For isotropic turbulence, the most general possible (isotropic) expression for $M_{ijk\ell}$ is

$$M_{ijk\ell} = k(\alpha\delta_{ij}\delta_{k\ell} + \beta\delta_{ik}\delta_{j\ell} + \gamma\delta_{i\ell}\delta_{jk}), \tag{1}$$

where α, β and γ are constants.
From Exercise 11.17, we know that $M_{ijk\ell}$ satisfies the following conditions:

$$M_{ijk\ell} = M_{jik\ell}, \quad M_{ijk\ell} = M_{ij\ell k}, M_{ijj\ell} = 0, \quad M_{ijkk} = \langle u_i u_j \rangle. \tag{2}$$

With $M_{ijk\ell} = M_{jik\ell}$, we get

$$k(\alpha\delta_{ij}\delta_{k\ell} + \beta\delta_{ik}\delta_{j\ell} + \gamma\delta_{i\ell}\delta_{jk}) = k(\alpha\delta_{ji}\delta_{k\ell} + \beta\delta_{jk}\delta_{i\ell} + \gamma\delta_{j\ell}\delta_{ik}). \tag{3}$$

So from Eq. 3, we get

$$\beta = \gamma. \tag{4}$$

With $M_{ijj\ell} = 0$, we get

$$k(\alpha\delta_{ij}\delta_{j\ell} + \beta\delta_{ij}\delta_{j\ell} + 3\gamma\delta_{i\ell}) = 0. \tag{5}$$

So form Eq. 5, we get

$$\alpha + \beta + 3\gamma = 0. \tag{6}$$

With $M_{ijkk} = \langle u_i u_j \rangle$, we get

$$\begin{aligned} M_{ijkk} &= k(\alpha\delta_{ij}\delta_{kk} + \beta\delta_{ik}\delta_{jk} + \gamma\delta_{ik}\delta_{jk}) \\ &= \langle u_i u_j \rangle = \frac{2}{3}k\delta_{ij}, \end{aligned} \tag{7}$$

where the last step follows from the isotropic turbulence.
So from Eq. 7, we get

$$3\alpha + \beta + \gamma = \frac{2}{3}. \tag{8}$$

With Eqs. 4, 6 and 8, we get

$$\alpha = \frac{4}{15}, \quad \beta = \gamma = -\frac{1}{15}. \tag{9}$$