Turbulent Flows
Stephen B. Pope
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Solution to Exercise 11.2

Prepared by: Christian Bauer *Date:* February 12, 2017

a) A solution to $\nabla^2 f(\mathbf{x}) = S(\mathbf{x})$ (2.44) is

$$f(\mathbf{x}) = \iiint_{\mathcal{V}} g(\mathbf{x}|\mathbf{y}) S(\mathbf{y}) d\mathbf{y} = \frac{-1}{4\pi} \iiint_{\mathcal{V}} \frac{S(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y}. \tag{2.48}$$

The rapid pressure $p^{(r)}$ satisfies

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}, \tag{11.11}$$

where $\partial\langle U_i\rangle/\partial x_j$is uniform due to homogeneous turbulence

$$\frac{1}{\rho}\nabla^2 p^{(r)}(\mathbf{x}) = -2\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j(\mathbf{x})}{\partial x_i}. \tag{1}$$

Substituting eqaution (11.11) into (2.48) gives the solution for the rapid pressure

$$\frac{1}{\rho}p^{(r)}(\mathbf{x}) = \frac{1}{2\pi}\iiint_{-\infty}^{\infty} \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j(\mathbf{y})}{\partial y_i}\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{2}$$

Multiplying both sides of equation (2) with a random field $\phi(\mathbf{x})$ gives

$$\begin{aligned}\frac{1}{\rho}p^{(r)}(\mathbf{x})\phi(\mathbf{x}) &= \frac{1}{2\pi}\iiint_{-\infty}^{\infty} \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j(\mathbf{y})}{\partial y_i}\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\phi(\mathbf{x}) \\ &= \frac{1}{2\pi}\frac{\partial\langle U_i\rangle}{\partial x_j}\iiint_{-\infty}^{\infty} \frac{\partial u_j(\mathbf{y})}{\partial y_i}\phi(\mathbf{x})\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \end{aligned} \tag{3}$$

Taking the average of equation 3 yields

$$\frac{1}{\rho}\langle p^{(r)}(\mathbf{x})\phi(\mathbf{x})\rangle = \left\langle \frac{1}{2\pi}\frac{\partial\langle U_i\rangle}{\partial x_j}\iiint_{-\infty}^{\infty}\frac{\partial u_j(\mathbf{y})}{\partial x_i}\phi(\mathbf{x})\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right\rangle$$

$$= \frac{1}{2\pi}\frac{\partial\langle U_k\rangle}{\partial x_l}\iiint_{-\infty}^{\infty}\left\langle\frac{\partial u_l(\mathbf{y})}{\partial y_k}\phi(\mathbf{x})\right\rangle\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \qquad (11.18)$$

b) For any integrand $T(\mathbf{x},\mathbf{y})$, the volume integral $I(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} T(\mathbf{x},\mathbf{y})d\mathbf{y}$ can be re-expressed as $I(\mathbf{x}) = \int_0^\infty \bar{T}(r)4\pi r^2\,dr$, with $r \equiv |\mathbf{x}-\mathbf{y}|$, and with $\bar{T}(r)$ denoting the the average of T over spherical shells of radius r, centered on $\mathbf{x}$. Hence for the integral I to converge, $\bar{T}(r)$ must tend to zero faster than r^{-3}. It follows therefore that, for the integral in Eq.(11.18) to converge, the shell-average of the two-point correlation has to tend to zero faster than r^{-2}.

c) The contribution of the rapid pressure to the pressure-rate-of-strain term is defined as

$$\mathcal{R}_{ij}^{(r)} = \left\langle\frac{p^{(r)}}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle \qquad (11.13)$$

$$= \left\langle\frac{p^{(r)}}{\rho}\frac{\partial u_i}{\partial x_j}\right\rangle + \left\langle\frac{p^{(r)}}{\rho}\frac{\partial u_j}{\partial x_i}\right\rangle. \qquad (4)$$

Now having $\phi(\mathbf{x})=\left(\frac{\partial u_i}{\partial x_j}\right)_{\mathbf{x}}$ equation (11.18) yields

$$\left\langle\frac{p^{(r)}}{\rho}\frac{\partial u_i}{\partial x_j}\right\rangle = \frac{1}{2\pi}\frac{\partial\langle U_k\rangle}{\partial x_l}\iiint_{-\infty}^{\infty}\left\langle\frac{\partial u_i(\mathbf{x})}{\partial x_j}\frac{\partial u_l(\mathbf{y})}{\partial y_k}\right\rangle\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \qquad (5)$$

where the term under the integral can be modified to

$$\left\langle \frac{\partial u_i(\mathbf{x})}{\partial x_j} \frac{\partial u_l(\mathbf{y})}{\partial y_k} \right\rangle = \left\langle \frac{\partial u_i(\mathbf{x})}{\partial x_j} \frac{\partial u_l(\mathbf{y})}{\partial y_k} + u_i(\mathbf{y}) \underbrace{\frac{\partial}{\partial x_j} \frac{\partial u_l(\mathbf{y})}{\partial y_k}}_{=0} \right\rangle$$
$$= \frac{\partial}{\partial x_j} \left\langle u_i(\mathbf{x}) \frac{\partial u_l(\mathbf{y})}{\partial y_k} \right\rangle$$
$$= \frac{\partial}{\partial x_j} \left\langle \underbrace{\frac{\partial u_i(\mathbf{x})}{\partial y_k}}_{=0} u_l(\mathbf{y}) + u_i(\mathbf{x}) \frac{\partial u_l(\mathbf{y})}{\partial y_k} \right\rangle$$
$$= \frac{\partial^2}{\partial x_j y_k} \left\langle u_i(\mathbf{x}) u_l(\mathbf{y}) \right\rangle . \tag{6}$$

Substituting equation 6 back into equation 5 yields

$$\left\langle \frac{p^{(r)}}{\rho} \frac{\partial u_i}{\partial x_j} \right\rangle = \frac{1}{2\pi} \frac{\partial \langle U_k \rangle}{\partial x_l} \iiint_{-\infty}^{\infty} \frac{\partial^2}{\partial x_j y_k} \langle u_i(\mathbf{x}) u_l(\mathbf{y}) \rangle \frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{11.19}$$

d) Substituting $\mathbf{y} = \mathbf{r} + \mathbf{x}$, $d\mathbf{y} = d\mathbf{r}$, $\partial y_i = \partial r_i$ and $\partial x_i = -\partial r_i$ into equation (11.19), with $\mathbf{r}$ the separation vector, yields

$$\left\langle \frac{p^{(r)}}{\rho} \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{2\pi} \frac{\partial \langle U_k \rangle}{\partial x_l} \iiint_{-\infty}^{\infty} \frac{\partial^2}{\partial r_j r_k} \langle u_i(\mathbf{x}) u_l(\mathbf{x}+\mathbf{r}) \rangle \frac{d\mathbf{r}}{|\mathbf{r}|}, \tag{7}$$

where $\langle u_i(\mathbf{x}) u_l(\mathbf{x}+\mathbf{r}) \rangle = R_{il}$ is the two-point velocity correlation

$$\left\langle \frac{p^{(r)}}{\rho} \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{2\pi} \frac{\partial \langle U_k \rangle}{\partial x_l} \iiint_{-\infty}^{\infty} \frac{1}{|\mathbf{r}|} \frac{\partial^2 R_{il}}{\partial r_j r_k} d\mathbf{r}. \tag{11.20}$$

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