

Turbulent Flows
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Solution to Exercise 11.3

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a) The definition of b_{ij} is

$$b_{ij} = \frac{\langle u_i u_j \rangle}{\langle u_k u_k \rangle} - \frac{1}{3} \delta_{ij} = \frac{\langle u_i u_j \rangle}{2k} - \frac{1}{3} \delta_{ij}. \quad (1)$$

And in this case, the exact Reynolds-stress equation is

$$\frac{d\langle u_i u_j \rangle}{dt} = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}. \quad (2)$$

So from Eq. 2, we get

$$\frac{d}{dt} \left(\frac{\langle u_i u_j \rangle}{2k} 2k \right) = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}, \quad (3)$$

and

$$\frac{d}{dt} \left(\frac{\langle u_i u_j \rangle}{2k} \right) 2k + \frac{\langle u_i u_j \rangle}{2k} 2 \frac{dk}{dt} = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}. \quad (4)$$

For this case, the exact equation for turbulent kinetic energy is

$$\frac{dk}{dt} = -\varepsilon. \quad (5)$$

Substituting Eq. 5 into Eq. 4 and using the following relation

$$\frac{\langle u_i u_j \rangle}{2k} = b_{ij} + \frac{1}{3} \delta_{ij}, \quad (6)$$

we get

$$\frac{d}{dt} \left(b_{ij} + \frac{1}{3} \delta_{ij} \right) 2k - \left(b_{ij} + \frac{1}{3} \delta_{ij} \right) 2\varepsilon = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}, \quad (7)$$

i.e.

$$\frac{db_{ij}}{dt} = \frac{\varepsilon}{k} \left(b_{ij} + \frac{1}{3}\delta_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\varepsilon} - \frac{\varepsilon_{ij}}{2\varepsilon} \right). \quad (8)$$

Hence the Eq.(11.23) follows from the assumed isotropy of ε_{ij} .

b) If ε_{ij} is taken to be proportional to $\langle u_i u_j \rangle$, i.e.

$$\varepsilon_{ij} = \frac{\varepsilon}{k} \langle u_i u_j \rangle, \quad (9)$$

where the coefficient ε/k follows from $\varepsilon_{ii} = 2\varepsilon$, then Eq. 8 becomes

$$\frac{db_{ij}}{dt} = \frac{\varepsilon}{k} \left(b_{ij} + \frac{1}{3}\delta_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\varepsilon} - \frac{\varepsilon \langle u_i u_j \rangle}{k 2\varepsilon} \right). \quad (10)$$

Substituting Eq. 1 into Eq. 10, we finally get

$$\frac{db_{ij}}{dt} = \frac{\mathcal{R}_{ij}^{(s)}}{2k}. \quad (11)$$

c) Rotta's model for $\mathcal{R}_{ij}^{(s)}$ is

$$\mathcal{R}_{ij}^{(s)} = -C_R \frac{\varepsilon}{k} \left(\langle u_i u_j \rangle - \frac{2}{3}k\delta_{ij} \right) = -2C_R \varepsilon b_{ij}. \quad (12)$$

Substituting Eq. 12 into Eq. 11, we get

$$\frac{db_{ij}}{dt} = -\frac{2C_R \varepsilon b_{ij}}{2k} = -C_R \frac{\varepsilon}{k} b_{ij}. \quad (13)$$

So the result is the same as Eq.(11.25) but with C_R in place of $C_R - 1$.

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