Turbulent Flows
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Solution to Exercise 11.3

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a) The definition of b_{ij} is

$$b_{ij} = \frac{\langle u_i u_j \rangle}{\langle u_k u_k \rangle} - \frac{1}{3}\delta_{ij} = \frac{\langle u_i u_j \rangle}{2k} - \frac{1}{3}\delta_{ij}. \tag{1}$$

And in this case, the exact Reynolds-stress equation is

$$\frac{\mathrm{d}\langle u_i u_j \rangle}{\mathrm{d}t} = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}. \tag{2}$$

So from Eq. 2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle u_i u_j \rangle}{2k} 2k\right) = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}, \tag{3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle u_i u_j \rangle}{2k}\right) 2k + \frac{\langle u_i u_j \rangle}{2k} 2\frac{\mathrm{d}k}{\mathrm{d}t} = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}. \tag{4}$$

For this case, the exact equation for turbulent kinetic energy is

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\varepsilon. \tag{5}$$

Substituting Eq. 5 into Eq. 4 and using the following relation

$$\frac{\langle u_i u_j \rangle}{2k} = b_{ij} + \frac{1}{3}\delta_{ij}, \tag{6}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(b_{ij} + \frac{1}{3}\delta_{ij}\right) 2k - \left(b_{ij} + \frac{1}{3}\delta_{ij}\right) 2\varepsilon = \mathcal{R}_{ij}^{(s)} - \varepsilon_{ij}, \tag{7}$$

i.e.

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = \frac{\varepsilon}{k}\left(b_{ij} + \frac{1}{3}\delta_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\varepsilon} - \frac{\varepsilon_{ij}}{2\varepsilon}\right). \tag{8}$$

Hence the Eq.(11.23) follows from the assumed isotropy of ε_{ij}.

b) If ε_{ij} is taken to be proportional to $\langle u_i u_j \rangle$, i.e.

$$\varepsilon_{ij} = \frac{\varepsilon}{k}\langle u_i u_j \rangle, \tag{9}$$

where the coefficient ε/k follows from $\varepsilon_{ii} = 2\varepsilon$, then Eq. 8 becomes

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = \frac{\varepsilon}{k}\left(b_{ij} + \frac{1}{3}\delta_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\varepsilon} - \frac{\varepsilon}{k}\frac{\langle u_i u_j \rangle}{2\varepsilon}\right). \tag{10}$$

Substituting Eq. 1 into Eq. 10, we finally get

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = \frac{\mathcal{R}_{ij}^{(s)}}{2k}. \tag{11}$$

c) Rotta's model for $\mathcal{R}_{ij}^{(s)}$ is

$$\mathcal{R}_{ij}^{(s)} = -C_R\frac{\varepsilon}{k}\left(\langle u_i u_j \rangle - \frac{2}{3}k\delta_{ij}\right) = -2C_R\varepsilon b_{ij}. \tag{12}$$

Substituting Eq. 12 into Eq. 11, we get

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} = -\frac{2C_R\varepsilon b_{ij}}{2k} = -C_R\frac{\varepsilon}{k}b_{ij}. \tag{13}$$

So the result is the same as Eq.(11.25) but with C_R in place of $C_R - 1$.