Turbulent Flows
Stephen B. Pope
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Solution to Exercise 12.23

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a): If the governing equation for U_i is

$$\frac{\mathrm{D}U_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i} + \bar{A}_i(\mathbf{U}, \mathbf{x}, t), \tag{1}$$

where $\bar{A}_i$ is a differentiable function, following from Eq. (12.8) on page 465, we get that the corresponding Eulerian PDF of $\mathbf{U}(\mathbf{x}, t)$, $f^*(\mathbf{V}; \mathbf{x}, t)$, evolves by

$$\frac{\partial f^*}{\partial t} + V_i \frac{\partial f^*}{\partial x_i} = -\frac{\partial}{\partial V_i}\left[f^* \left(-\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i} + \bar{A}_i(\mathbf{V}, \mathbf{x}, t) \right) \right], \tag{2}$$

i.e.,

$$\frac{\partial f^*}{\partial t} + V_i \frac{\partial f^*}{\partial x_i} - \frac{1}{\rho}\frac{\partial f^*}{\partial V_i}\frac{\partial \langle p \rangle}{\partial x_i} = -\frac{\partial}{\partial V_i}[f^* \bar{A}_i(\mathbf{V}, \mathbf{x}, t)]. \tag{3}$$

b): From Eq.(12.117), we get that the Eulerian PDF equation stemming from the GLM is

$$\begin{aligned} \frac{\partial f^*}{\partial t} + V_i \frac{\partial f^*}{\partial x_i} - \frac{1}{\rho}\frac{\partial f^*}{\partial V_i}\frac{\partial \langle p \rangle}{\partial x_i} &= -G_{ij}\frac{\partial}{\partial V_i}\left[f^* \left(V_j - \langle U_j \rangle \right) \right] \\ &+ \tfrac{1}{2} C_0 \varepsilon \frac{\partial^2 f^*}{\partial V_i \partial V_i}. \end{aligned} \tag{4}$$

Equation 4 can be rewritten as

$$\frac{\partial f^*}{\partial t} + V_i \frac{\partial f^*}{\partial x_i} - \frac{1}{\rho}\frac{\partial f^*}{\partial V_i}\frac{\partial \langle p \rangle}{\partial x_i} = -\frac{\partial}{\partial V_i}\left[G_{ij} f^* \left(V_j - \langle U_j \rangle \right) - \tfrac{1}{2} C_0 \varepsilon \frac{\partial f^*}{\partial V_i} \right], \tag{5}$$

i.e.,

$$\frac{\partial f^*}{\partial t}+V_i\frac{\partial f^*}{\partial x_i}-\frac{1}{\rho}\frac{\partial f^*}{\partial V_i}\frac{\partial \langle p\rangle}{\partial x_i} = -\frac{\partial}{\partial V_i}\left[f^*\left(G_{ij}\left(V_j-\langle U_j\rangle\right)-\tfrac{1}{2}C_0\varepsilon\frac{\partial \ln f^*}{\partial V_i}\right)\right]. \tag{6}$$

So the Eulerian PDF equation stemming from the GLM can be written in the same form as Eq. 3 with

$$\bar{A}_i = G_{ij}(V_j-\langle U_j\rangle)-\tfrac{1}{2}C_0\varepsilon\frac{\partial \ln f^*}{\partial V_i}. \tag{7}$$

c): If f^* is joint normal, then f^* can be expressed as

$$f^*(\mathbf{V}) = \left[(2\pi)^3\det(\mathbf{C})\right]^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}(\mathbf{V}-\langle\mathbf{U}\rangle)^{\mathbf{T}}\mathbf{C^{-1}}(\mathbf{V}-\langle\mathbf{U}\rangle)\right], \tag{8}$$

where $\mathbf{C}$ is the covariance matrix ($\mathbf{C}=\mathbf{u}\mathbf{u}^T$) and which is symmetric. So

$$\begin{aligned}
\frac{\partial \ln f^*}{\partial V_i} &= \frac{\partial}{\partial V_i}\left\{\ln\left[(2\pi)^3\det(\mathbf{C})\right]^{-\frac{1}{2}}-\tfrac{1}{2}(V_k-\langle U_k\rangle)C^{-1}_{kj}(V_j-\langle U_j\rangle)\right\}\\
&= -\tfrac{1}{2}C^{-1}_{kj}(V_j-\langle U_j\rangle)\delta_{ki}-\tfrac{1}{2}(V_k-\langle U_k\rangle)C^{-1}_{kj}\delta_{ij}\\
&= -\tfrac{1}{2}C^{-1}_{ij}(V_j-\langle U_j\rangle)-\tfrac{1}{2}C^{-1}_{ki}(V_k-\langle U_k\rangle)\\
&= -\tfrac{1}{2}C^{-1}_{ij}(V_j-\langle U_j\rangle)-\tfrac{1}{2}C^{-1}_{ik}(V_k-\langle U_k\rangle)\\
&= -C^{-1}_{ij}(V_j-\langle U_j\rangle).
\end{aligned} \tag{9}$$

Then $\bar{\mathbf{A}}$ is given by

$$\begin{aligned}
\bar{A}_i &= G_{ij}(V_j-\langle U_j\rangle)-\tfrac{1}{2}C_0\varepsilon\frac{\partial \ln f^*}{\partial V_i}\\
&= (G_{ij}+\tfrac{1}{2}C_0\varepsilon C^{-1}_{ij})(V_j-\langle U_j\rangle),
\end{aligned} \tag{10}$$

where C^{-1}_{ij} is the i-j component of the inverse of the Reynolds-stress tensor $C_{ij}=\langle u_i u_j\rangle$.

So, for homogeneous turbulence, in which f^* is joint-normal, Eq. 1 and Eq. 10 provide a deterministic model with the same Eulerian PDF evolution as the GLM, because both the deterministic model and the GLM model preserve the joint-normal PDF, provided that the initial PDF is joint normal and the evolution equations of the Eulerian PDF for both models are the same.

For a general flow, because the deterministic model preserves the PDF shape (as a result of lacking a diffusion term), the resulting PDF does not relax to joint-normal.