Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 12.24

Prepared by: Zhuyin Ren *Date:* 06/05/03

From the Navier-Stokes equations (Eq.(2.35)), we get

$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{U}. \tag{1}$$

Following a fluid particle, from Navier-Stokes equations we get

$$\frac{\mathrm{d}U_i^*}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 U_i^*. \tag{2}$$

From the GLM equation (Eq.(12.110)), we get

$$\mathrm{d}U_i^*(t) = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}\,\mathrm{d}t + G_{ij}(U_j^*(t) - \langle U_j\rangle)\,\mathrm{d}t + (C_0\varepsilon)^{\frac{1}{2}}\,\mathrm{d}W_i(t), \tag{3}$$

where the coefficient $G_{ij}(\mathbf{x}, t)$ (which is evaluated at $\mathbf{X}^*(t)$) depends on the local values of $\langle u_i u_j\rangle$, ε and $\partial\langle U_i\rangle/\partial x_j$.
From Eq. 3, we get

$$\frac{\mathrm{d}U_i^*}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i} + G_{ij}(U_j^* - \langle U_j\rangle) + (C_0\varepsilon)^{\frac{1}{2}}\dot{W}_i, \tag{4}$$

where $\dot{\mathbf{W}}$ is white noise (see Eq. (J.29)).
Subtracting Eq. 2 from Eq. 4, we get

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 U_i^* = G_{ij}(U_j^* - \langle U_j\rangle) + (C_0\varepsilon)^{\frac{1}{2}}\dot{W}_i. \tag{5}$$

So the Generalized Langevin Model amounts to modelling the fluctuating pressure gradient and viscous terms as

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 U_i = \tilde{A}_i(\mathbf{U}, \mathbf{x}, t) \equiv G_{ij}(U_j - \langle U_j\rangle) + (C_0\varepsilon)^{\frac{1}{2}}\dot{W}_i. \tag{6}$$