Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 12.25

Prepared by: Zhuyin Ren *Date:* 06/08/03

a): According to the definition of the fluctuating velocity following a particle, we get

$$\mathbf{u}^*(t) \equiv \mathbf{U}^*(t) - \langle \mathbf{U}(\mathbf{X}^*[t], t)\rangle. \tag{1}$$

Supposing $\mathbf{X}^*(t)$ evolves by

$$\frac{\mathrm{d}\mathbf{X}^*(t)}{\mathrm{d}t} = \mathbf{U}^*(t), \tag{2}$$

and $\mathbf{U}^*(\mathbf{t})$ evolves by a diffusion process, from Eq. 1, we get

$$\begin{aligned}
\mathrm{d}u_i^*(t) &= \mathrm{d}U_i^*(t) - \mathrm{d}\langle U_i(\mathbf{X}^*[t], t)\rangle \\
&= \mathrm{d}U_i^*(t) - \left[\frac{\partial \langle U_i\rangle}{\partial t} + \left(\frac{\partial \langle U_i\rangle}{\partial x_j}\right)_{\mathbf{x}=\mathbf{X}^*} \frac{\mathrm{d}X_j^*}{\mathrm{d}t}\right] \mathrm{d}t \\
&= \mathrm{d}U_i^*(t) - \left(\frac{\partial \langle U_i\rangle}{\partial t} + \frac{\mathrm{d}X_j^*}{\mathrm{d}t}\frac{\partial \langle U_i\rangle}{\partial x_j}\right) \mathrm{d}t \\
&= \mathrm{d}U_i^*(t) - \left[\frac{\partial \langle U_i\rangle}{\partial t} + \left(u_j^*(t) + \langle U_j\rangle\right)\frac{\partial \langle U_i\rangle}{\partial x_j}\right] \mathrm{d}t \\
&= \mathrm{d}U_i^*(t) - u_j^* \frac{\partial \langle U_i\rangle}{\partial x_j}\,\mathrm{d}t - \frac{\bar{\mathrm{D}}\langle U_i\rangle}{\bar{\mathrm{D}}t}\,\mathrm{d}t.
\end{aligned} \tag{3}$$

b): For homogeneous turbulence, and for $\mathbf{U}^*(t)$ evolving by (GLM)

$$\mathrm{d}U_i^*(t) = -\frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\,\mathrm{d}t + G_{ij}(U_j^*(t) - \langle U_j\rangle)\,\mathrm{d}t + (C_0\varepsilon)^{\frac{1}{2}}\,\mathrm{d}W_i(t), \tag{4}$$

we get

$$\mathrm{d}u_i^*(t) = -\frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\,\mathrm{d}t + G_{ij}(U_j^*(t) - \langle U_j\rangle)\,\mathrm{d}t + (C_0\varepsilon)^{\frac{1}{2}}\,\mathrm{d}W_i(t)$$

$$
\begin{aligned}
& -u_j^* \frac{\partial \langle U_i \rangle}{\partial x_j}\,\mathrm{d}t - \frac{\bar{\mathrm{D}}\langle U_i\rangle}{\bar{\mathrm{D}}t}\,\mathrm{d}t \\
= & -\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}\,\mathrm{d}t + G_{ij}(U_j^*(t) - \langle U_j \rangle)\,\mathrm{d}t + (C_0\varepsilon)^{\frac{1}{2}}\,\mathrm{d}W_i(t) \\
& -u_j^* \frac{\partial \langle U_i \rangle}{\partial x_j}\,\mathrm{d}t + \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}\,\mathrm{d}t - \nu\nabla^2\langle U_i\rangle\,\mathrm{d}t \\
= & -u_j^* \frac{\partial \langle U_i \rangle}{\partial x_j}\,\mathrm{d}t + G_{ij}u_j^*\,\mathrm{d}t + (C_0\varepsilon)^{\frac{1}{2}}\,\mathrm{d}W_i(t),
\end{aligned}
\tag{5}
$$

where the second step follows from the mean momentum equation and the last step follows from homogeneity.

c): According to the Fokker-Plank equation, the evolution equation for $g_L^*(\mathbf{v}, \mathbf{x}; t|\mathbf{Y})$ can be derived from Eq 2 and Eq. 5. The result is

$$
\begin{aligned}
\frac{\partial g_L^*}{\partial t} + \frac{\partial}{\partial x_i}\left[g_L^*\left(\langle U_i\rangle + v_i\right)\right] &= \frac{\partial}{\partial v_i}\left[g_L^* v_j \frac{\partial\langle U_i\rangle}{\partial x_j}\right] - \frac{\partial}{\partial v_i}\left[G_{ij}v_j g_L^*\right] \\
&\quad + \tfrac{1}{2}(C_0\varepsilon)\frac{\partial^2 g_L^*}{\partial v_i \partial v_i}.
\end{aligned}
\tag{6}
$$

Using $\nabla \cdot \langle \mathbf{U} \rangle = 0$, form Eq. 6, we get

$$
\begin{aligned}
\frac{\partial g_L^*}{\partial t} + \left(\langle U_i\rangle + v_i\right)\frac{\partial g_L^*}{\partial x_i} &= v_j \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial g_L^*}{\partial v_i} - \frac{\partial}{\partial v_i}\left[G_{ij}v_j g_L^*\right] \\
&\quad + \tfrac{1}{2}(C_0\varepsilon)\frac{\partial^2 g_L^*}{\partial v_i \partial v_i}.
\end{aligned}
\tag{7}
$$

Integrating Eq. 7 for g_L^* over all $\mathbf{Y}$, we get the evolution equation for $g^*(\mathbf{x}; t)$

$$
\begin{aligned}
\frac{\partial g^*}{\partial t} + \left(\langle U_i\rangle + v_i\right)\frac{\partial g^*}{\partial x_i} &= v_j \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial g^*}{\partial v_i} - \frac{\partial}{\partial v_i}\left[G_{ij}v_j g^*\right] \\
&\quad + \tfrac{1}{2}(C_0\varepsilon)\frac{\partial^2 g^*}{\partial v_i \partial v_i}.
\end{aligned}
\tag{8}
$$

For homogeneous turbulence, the second term on the left-hand side of Eq. 8 vanishes, so Eq. 8 turns into

$$
\frac{\partial g^*}{\partial t} - v_j \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial g^*}{\partial v_i} = -\frac{\partial}{\partial v_i}\left[G_{ij}v_j g^*\right] + \tfrac{1}{2}(C_0\varepsilon)\frac{\partial^2 g^*}{\partial v_i \partial v_i}, \tag{9}
$$

and this is identical to Eq.(12.27).

d): According to Eq. 9, we get

$$\frac{\partial g^*}{\partial t} = v_j \frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial g^*}{\partial v_i} - \frac{\partial}{\partial v_i} \left[G_{ij} v_j g^* \right] + \tfrac{1}{2}(C_0 \varepsilon) \frac{\partial^2 g^*}{\partial v_i \partial v_i}. \tag{10}$$

Multiplying Eq. 10 by $v_i v_j$ and integrating over the fluctuating velocity space, we get

$$\begin{aligned} \int v_i v_j \frac{\partial g^*}{\partial t} \mathrm{d}\mathbf{v} &= \int v_i v_j v_k \frac{\partial \langle U_\ell \rangle}{\partial x_k} \frac{\partial g^*}{\partial v_\ell} \mathrm{d}\mathbf{v} - \int v_i v_j \frac{\partial}{\partial v_k} \left(G_{k\ell} v_\ell g^* \right) \mathrm{d}\mathbf{v} \\ &+ \int v_i v_j \tfrac{1}{2} C_0 \varepsilon \frac{\partial^2 g^*}{\partial v_k \partial v_k} \mathrm{d}\mathbf{v}. \end{aligned} \tag{11}$$

From Eq. 11, it is easy to get

$$\begin{aligned} \frac{\partial \langle u_i^* u_j^* \rangle}{\partial t} &= -\langle u_j^* u_k^* \rangle \frac{\partial \langle U_i \rangle}{\partial x_k} - \langle u_i^* u_k^* \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} + G_{j\ell} \langle u_\ell^* u_i^* \rangle \\ &+ G_{i\ell} \langle u_\ell^* u_j^* \rangle + C_0 \varepsilon \delta_{ij}, \end{aligned} \tag{12}$$

and this is consistent with Eq.(12.57). In particular, the first term on the right-hand side of Eq. 5 leads to the production.