

**Turbulent Flows**  
Stephen B. Pope  
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**Solution to Exercise 12.35**

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In Eq.(3.68), the normalized raw moments of a random variable  $U$  with sample space variable  $V$  and mean  $\mu$  are given. Substituting  $U$  by  $\omega$  and  $\mu$  by  $\langle\omega\rangle$  leads

$$\frac{\langle\omega^n\rangle}{\langle\omega\rangle^n} = \frac{\Gamma(n+\alpha)}{\alpha^n\Gamma(\alpha)} = \frac{(n+\alpha-1)!}{\alpha^n(\alpha-1)!}. \quad (1)$$

Assuming that the comparison between the moments of  $\omega$  and the log-normally distributed random variable  $\varepsilon_r$  shall be made for equal normalized second moments, we get for  $\omega$  from Eq.(1) with  $n = 2$

$$\frac{\langle\omega^2\rangle}{\langle\omega\rangle^2} = \frac{(2+\alpha-1)!}{\alpha^2(\alpha-1)!} = \frac{(\alpha+1)!}{\alpha\alpha!} = \frac{\alpha+1}{\alpha} \quad (2)$$

and for  $\varepsilon_r$  from Eq.(6.333)

$$\frac{\langle\varepsilon_r^2\rangle}{\langle\varepsilon_r\rangle^2} = \exp[\tfrac{1}{2}\sigma^2 2(2-1)] = \exp(\sigma^2). \quad (3)$$

Setting Eq.(2) equal to Eq.(3) leads a relation for the parameter  $\sigma$ ,

$$\exp(\sigma^2) = \frac{\alpha+1}{\alpha}. \quad (4)$$

Using this result in Eq.(6.333) gives

$$\frac{\langle\varepsilon_r^n\rangle}{\langle\varepsilon_r\rangle^n} = \exp[\tfrac{1}{2}\sigma^2 n(n-1)] = \exp(\sigma^2)^{\frac{n(n-1)}{2}} = \left(\frac{\alpha+1}{\alpha}\right)^{\frac{n(n-1)}{2}}. \quad (5)$$

The third raw moments ( $n = 3$ ) of  $\omega$  (Eq.(1)) and  $\varepsilon_r$  (Eq.(5)) are then

$$\begin{aligned} \frac{\langle\omega^3\rangle}{\langle\omega\rangle^3} &= \frac{(\alpha+2)!}{\alpha^2\alpha!} = \frac{(\alpha+2)(\alpha+1)}{\alpha^2} \quad \text{and} \\ \frac{\langle\varepsilon_r^3\rangle}{\langle\varepsilon_r\rangle^3} &= \left(\frac{\alpha+1}{\alpha}\right)^3, \end{aligned} \quad (6)$$

respectively. Similarly, for the fourth raw moment ( $n = 4$ )

$$\begin{aligned}\frac{\langle \omega^4 \rangle}{\langle \omega \rangle^4} &= \frac{(\alpha + 3)!}{\alpha^3 \alpha!} = \frac{(\alpha + 3)(\alpha + 2)(\alpha + 1)}{\alpha^3} \quad \text{and} \\ \frac{\langle \varepsilon_r^4 \rangle}{\langle \varepsilon_r \rangle^4} &= \left( \frac{\alpha + 1}{\alpha} \right)^6.\end{aligned}\tag{7}$$

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