Turbulent Flows
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Solution to Exercise 12.35

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In Eq.(3.68), the normalized raw moments of a random variable U with sample space variable V and mean μ are given. Substituting U by ω and μ by $\langle\omega\rangle$ leads

$$\frac{\langle\omega^n\rangle}{\langle\omega\rangle^n} = \frac{\Gamma(n+\alpha)}{\alpha^n\Gamma(\alpha)} = \frac{(n+\alpha-1)!}{\alpha^n(\alpha-1)!}. \tag{1}$$

Assuming that the comparison between the moments of ω and the log-normally distributed random variable ε_r shall be made for equal normalized second moments, we get for ω from Eq.(1) with $n = 2$

$$\frac{\langle\omega^2\rangle}{\langle\omega\rangle^2} = \frac{(2+\alpha-1)!}{\alpha^2(\alpha-1)!} = \frac{(\alpha+1)!}{\alpha\,\alpha!} = \frac{\alpha+1}{\alpha} \tag{2}$$

and for ε_r from Eq.(6.333)

$$\frac{\langle\varepsilon_r^2\rangle}{\langle\varepsilon_r\rangle^2} = \exp[\tfrac{1}{2}\sigma^2 2(2-1)] = \exp(\sigma^2). \tag{3}$$

Setting Eq.(2) equal to Eq.(3) leads a relation for the parameter σ,

$$\exp(\sigma^2) = \frac{\alpha+1}{\alpha}. \tag{4}$$

Using this result in Eq.(6.333) gives

$$\frac{\langle\varepsilon_r^n\rangle}{\langle\varepsilon_r\rangle^n} = \exp[\tfrac{1}{2}\sigma^2 n(n-1)] = \exp(\sigma^2)^{\frac{n(n-1)}{2}} = \left(\frac{\alpha+1}{\alpha}\right)^{\frac{n(n-1)}{2}}. \tag{5}$$

The third raw moments ($n = 3$) of ω (Eq.(1)) and ε_r (Eq.(5)) are then

$$\frac{\langle\omega^3\rangle}{\langle\omega\rangle^3} = \frac{(\alpha+2)!}{\alpha^2\alpha!} = \frac{(\alpha+2)(\alpha+1)}{\alpha^2} \quad \text{and}$$

$$\frac{\langle\varepsilon_r^3\rangle}{\langle\varepsilon_r\rangle^3} = \left(\frac{\alpha+1}{\alpha}\right)^3, \tag{6}$$

respectively. Similarly, for the fourth raw moment ($n = 4$)

$$\frac{\langle \omega^4 \rangle}{\langle \omega \rangle^4} = \frac{(\alpha+3)!}{\alpha^3 \alpha!} = \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{\alpha^3} \quad \text{and}$$

$$\frac{\langle \varepsilon_r^4 \rangle}{\langle \varepsilon_r \rangle^4} = \left(\frac{\alpha+1}{\alpha}\right)^6. \tag{7}$$