Turbulent Flows
Stephen B. Pope
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Solution to Exercise 12.37

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a) From the definitions Eqs.(12.204)-(12.206), we get the marginal PDF of position is

$$f^*_{\mathbf{X}}(\mathbf{x};t) = \int f^*_L(\mathbf{V},\mathbf{x};t)\,\mathrm{d}\mathbf{V}, \tag{1}$$

and the PDF of $\mathbf{U}^*(t)$ conditional on $\mathbf{X}^*(t)=\mathbf{x}$ is

$$f^*(\mathbf{V}|\mathbf{x};t) = f^*_L(\mathbf{V},\mathbf{x};t)/f^*_{\mathbf{X}}(\mathbf{x};t). \tag{2}$$

The first moment of this PDF is the conditional mean particle velocity

$$\langle \mathbf{U}^*(t) \mid \mathbf{X}^*(t)=\mathbf{x}\rangle = \int \mathbf{V} f^*(\mathbf{V}|\mathbf{x};t)\,\mathrm{d}\mathbf{V}, \tag{3}$$

which is abbreviated to $\langle \mathbf{U}^*|\mathbf{x}\rangle$.

So multiplying f^*_L with $\mathbf{V}$ and integrating over all $\mathbf{V}$, we get

$$\begin{aligned}\int \mathbf{V} f^*_L(\mathbf{V},\mathbf{x};t)\,\mathrm{d}\mathbf{V} &= \int \mathbf{V} f^*(\mathbf{V}|\mathbf{x};t) f^*_{\mathbf{X}}(\mathbf{x};t)\,\mathrm{d}\mathbf{V} \\ &= f^*_{\mathbf{X}}(\mathbf{x};t)\langle \mathbf{U}^*(t)|\mathbf{x}\rangle. \end{aligned}\tag{4}$$

b) The Fokker-Planck equation for $f^*_L(\mathbf{V},\mathbf{x};t)$ is

$$\frac{\partial f^*_L}{\partial t} + V_i\frac{\partial f^*_L}{\partial x_i} = -\frac{\partial}{\partial V_i}[f^*_L a_i(\mathbf{V},\mathbf{x},t)] + \tfrac{1}{2}b(\mathbf{x},t)^2\frac{\partial^2 f^*_L}{\partial V_i \partial V_i}. \tag{5}$$

By integrating the Fokker-Planck equation over all $\mathbf{V}$, we get

$$\frac{\partial \int f^*_L \mathrm{d}\mathbf{V}}{\partial t} + \int V_i\frac{\partial f^*_L}{\partial x_i}\mathrm{d}\mathbf{V} = \int -\frac{\partial}{\partial V_i}[f^*_L a_i(\mathbf{V},\mathbf{x},t)] + \tfrac{1}{2}b(\mathbf{x},t)^2\frac{\partial^2 f^*_L}{\partial V_i \partial V_i}\mathrm{d}\mathbf{V}, \tag{6}$$

i.e.,

$$\frac{\partial \int f_L^* \mathrm{d}\mathbf{V}}{\partial t} + \frac{\partial \int V_i f_L^* \mathrm{d}\mathbf{V}}{\partial x_i} = \int -\frac{\partial}{\partial V_i}[f_L^* a_i(\mathbf{V}, \mathbf{x}, t)] + \tfrac{1}{2} b(\mathbf{x}, t)^2 \frac{\partial^2 f_L^*}{\partial V_i \partial V_i} \mathrm{d}\mathbf{V}. \tag{7}$$

With the results of Exercise 12.1, the right side of Eq. 7 is zero. Substituting Eqs. 1 and 4 into Eq. 7, we get

$$\frac{\partial f_{\mathbf{X}}^*}{\partial t} + \frac{\partial}{\partial x_i}\left[\langle U_i^* | \mathbf{x} \rangle f_{\mathbf{X}}^*\right] = 0. \tag{8}$$

Divided Eq. 8 by $f_{\mathbf{X}}^*$, we get

$$\frac{1}{f_{\mathbf{X}}^*}\frac{\partial f_{\mathbf{X}}^*}{\partial t} + \langle U_i^* | \mathbf{x} \rangle \frac{1}{f_{\mathbf{X}}^*}\frac{\partial f_{\mathbf{X}}^*}{\partial x_i} + \frac{\partial \langle U_i^* | \mathbf{x} \rangle}{\partial x_i} = 0, \tag{9}$$

i.e.,

$$\left(\frac{\partial}{\partial t} + \langle U_i^* | \mathbf{x} \rangle \frac{\partial}{\partial x_i}\right) \ln f_{\mathbf{X}}^* = -\frac{\partial \langle U_i^* | \mathbf{x} \rangle}{\partial x_i}, \tag{10}$$

hence verifying Eq.(12.217).

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