Turbulent Flows
Stephen B. Pope
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Solution to Exercise 12.4

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The exact transport equation for $f(\mathbf{V};\mathbf{x},t)$ is

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\frac{\partial f}{\partial V_i} - \frac{\partial}{\partial V_i}\left[f \left\langle \nu \nabla^2 U_i - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} \,\Big|\, \mathbf{V} \right\rangle \right]. \tag{1}$$

Multiplying Eq. 1 by $v_j v_k$ and integrating over velocity space, where $\mathbf{v}$ is defined by

$$\mathbf{v}\,(\mathbf{V},\mathbf{x},t) \equiv \mathbf{V} - \langle \mathbf{U}(\mathbf{x},t)\rangle, \tag{2}$$

we get

$$\begin{aligned}
&\int \left(v_j v_k \frac{\partial f}{\partial t} + v_j v_k V_i \frac{\partial f}{\partial x_i} \right) \mathrm{d}\mathbf{V} \\
&= \int \left(\frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\frac{\partial f}{\partial V_i} - \frac{\partial}{\partial V_i}\left[f \left\langle \nu \nabla^2 U_i - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} \,\Big|\, \mathbf{V} \right\rangle \right] \right) v_j v_k \mathrm{d}\mathbf{V}.
\end{aligned} \tag{3}$$

a)

$$\begin{aligned}
\int v_j v_k \frac{\partial f}{\partial t} \mathrm{d}\mathbf{V} &= \int (V_j - \langle U_j\rangle)(V_k - \langle U_k\rangle) \frac{\partial f}{\partial t}\mathrm{d}\mathbf{V} \\
&= \int V_j V_k \frac{\partial f}{\partial t}\mathrm{d}\mathbf{V} + \int \langle U_j\rangle\langle U_k\rangle \frac{\partial f}{\partial t}\mathrm{d}\mathbf{V} \\
&\quad - \int V_j \langle U_k\rangle \frac{\partial f}{\partial t}\mathrm{d}\mathbf{V} - \int V_k \langle U_j\rangle \frac{\partial f}{\partial t}\mathrm{d}\mathbf{V} \\
&= \frac{\partial \langle U_j U_k\rangle}{\partial t} + 0 - \langle U_k\rangle \frac{\partial \langle U_j\rangle}{\partial t} - \langle U_j\rangle \frac{\partial \langle U_k\rangle}{\partial t} \\
&= \frac{\partial \langle U_j U_k\rangle}{\partial t} - \frac{\partial (\langle U_j\rangle\langle U_k\rangle)}{\partial t} \\
&= \frac{\partial \langle u_j u_k\rangle}{\partial t}.
\end{aligned} \tag{4}$$

b)

$$\begin{aligned}
v_j v_k V_i \frac{\partial f}{\partial x_i} &= \frac{\partial(v_j v_k V_i f)}{\partial x_i} - f\frac{\partial(v_j v_k V_i)}{\partial x_i} \\
&= \frac{\partial(v_j v_k V_i f)}{\partial x_i} - fV_i \frac{\partial\left[(V_j - \langle U_j\rangle)(V_k - \langle U_k\rangle)\right]}{\partial x_i} \\
&= \frac{\partial(v_j v_k V_i f)}{\partial x_i} + fV_i(V_j - \langle U_j\rangle)\frac{\partial\langle U_k\rangle}{\partial x_i} + fV_i(V_k - \langle U_k\rangle)\frac{\partial\langle U_j\rangle}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}\left(V_i v_j v_k f\right) + fV_i\left(v_k\frac{\partial\langle U_j\rangle}{\partial x_i} + v_j\frac{\partial\langle U_k\rangle}{\partial x_i}\right). \qquad (5)
\end{aligned}$$

So

$$\begin{aligned}
\int v_j v_k V_i \frac{\partial f}{\partial x_i}\mathrm{d}\mathbf{V} &= \int \left[\frac{\partial}{\partial x_i}\left(V_i v_j v_k f\right) + fV_i\left(v_k\frac{\partial\langle U_j\rangle}{\partial x_i} + v_j\frac{\partial\langle U_k\rangle}{\partial x_i}\right)\right]\mathrm{d}\mathbf{V} \\
&= \int \frac{\partial}{\partial x_i}\left[\left(\langle U_i\rangle + v_i\right) v_j v_k f\right]\mathrm{d}\mathbf{V} \\
&\quad + \int f(\langle U_i\rangle + v_i)\left(v_k\frac{\partial\langle U_j\rangle}{\partial x_i} + v_j\frac{\partial\langle U_k\rangle}{\partial x_i}\right)\mathrm{d}\mathbf{V} \\
&= \frac{\partial}{\partial x_i}\left(\langle U_i\rangle\langle u_j u_k\rangle + \langle u_i u_j u_k\rangle\right) + \langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_i} + \langle u_i u_j\rangle\frac{\partial\langle U_k\rangle}{\partial x_i} \\
&= \langle U_i\rangle\frac{\partial\langle u_j u_k\rangle}{\partial x_i} + \frac{\partial\langle u_i u_j u_k\rangle}{\partial x_i} + \langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_i} + \langle u_i u_j\rangle\frac{\partial\langle U_k\rangle}{\partial x_i} \qquad (6)
\end{aligned}$$

So the convection term accounts for mean convection, turbulent transport, and production.

c)

$$\begin{aligned}
\int \frac{1}{\rho} v_j v_k \frac{\partial\langle p\rangle}{\partial x_i}\frac{\partial f}{\partial V_i}\mathrm{d}\mathbf{V} &= \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}\int (V_j - \langle U_j\rangle)(V_k - \langle U_k\rangle)\frac{\partial f}{\partial V_i}\mathrm{d}\mathbf{V} \\
&= -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}\int f\frac{\partial}{\partial V_i}\left[(V_j - \langle U_j\rangle)(V_k - \langle U_k\rangle)\right]\mathrm{d}\mathbf{V} \\
&= -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}\int f\delta_{ij}(V_k - \langle U_k\rangle) + f\delta_{ik}(V_j - \langle U_j\rangle)\mathrm{d}\mathbf{V} \\
&= 0. \qquad (7)
\end{aligned}$$

So the mean pressure gradient term vanishes.

d)

$$
\begin{aligned}
&\int -v_j v_k \frac{\partial}{\partial V_i}\left[f\left\langle \nu\nabla^2 U_i \,\middle|\, \mathbf{V}\right\rangle\right]\mathrm{d}\mathbf{V} \\
&= \int f\left\langle \nu\nabla^2 U_i \,\middle|\, \mathbf{V}\right\rangle \frac{\partial}{\partial V_i}\left[(V_j-\langle U_j\rangle)(V_k-\langle U_k\rangle)\right]\mathrm{d}\mathbf{V} \\
&= \int f\left\langle \nu\nabla^2 U_i \,\middle|\, \mathbf{V}\right\rangle \left[(V_k-\langle U_k\rangle)\delta_{ij}+(V_j-\langle U_j\rangle)\delta_{ki}\right]\mathrm{d}\mathbf{V} \\
&= \int f\left\langle \nu\nabla^2 U_j \,\middle|\, \mathbf{V}\right\rangle (V_k-\langle U_k\rangle)\mathrm{d}\mathbf{V} + \int f\left\langle \nu\nabla^2 U_k \,\middle|\, \mathbf{V}\right\rangle (V_j-\langle U_j\rangle)\mathrm{d}\mathbf{V} \\
&= \langle u_k\nu\nabla^2 U_j\rangle + \langle u_j\nu\nabla^2 U_k\rangle \\
&= \langle u_k\nu\nabla^2 u_j\rangle + \langle u_j\nu\nabla^2 u_k\rangle \\
&= \nu\nabla^2\langle u_k u_j\rangle - 2\nu\langle\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_i}\rangle. \qquad (8)
\end{aligned}
$$

e)

$$
\begin{aligned}
&\int v_j v_k \frac{\partial}{\partial V_i}\left[f\left\langle \frac{1}{\rho}\frac{\partial p'}{\partial x_i} \,\middle|\, \mathbf{V}\right\rangle\right]\mathrm{d}\mathbf{V} \\
&= -\int f\left\langle \frac{1}{\rho}\frac{\partial p'}{\partial x_i} \,\middle|\, \mathbf{V}\right\rangle \frac{\partial}{\partial V_i}\left[(V_j-\langle U_j\rangle)(V_k-\langle U_k\rangle)\right]\mathrm{d}\mathbf{V} \\
&= -\int f\left\langle \frac{1}{\rho}\frac{\partial p'}{\partial x_i} \,\middle|\, \mathbf{V}\right\rangle \left[(V_k-\langle U_k\rangle)\delta_{ij}+(V_j-\langle U_j\rangle)\delta_{ki}\right]\mathrm{d}\mathbf{V} \\
&= -\int f\left\langle \frac{1}{\rho}\frac{\partial p'}{\partial x_j} \,\middle|\, \mathbf{V}\right\rangle (V_k-\langle U_k\rangle)\mathrm{d}\mathbf{V} - \int f\left\langle \frac{1}{\rho}\frac{\partial p'}{\partial x_k} \,\middle|\, \mathbf{V}\right\rangle (V_j-\langle U_j\rangle)\mathrm{d}\mathbf{V} \\
&= -\left\langle \frac{1}{\rho}u_k\frac{\partial p'}{\partial x_j}\right\rangle - \left\langle \frac{1}{\rho}u_j\frac{\partial p'}{\partial x_k}\right\rangle \\
&= -\frac{1}{\rho}\left\langle u_k\frac{\partial p'}{\partial x_j} + u_j\frac{\partial p'}{\partial x_k}\right\rangle. \qquad (9)
\end{aligned}
$$

Substituting Eqs. 4, 6, 7, 8 and 9 into Eq. 3, we get

$$
\begin{aligned}
&\frac{\partial\langle u_j u_k\rangle}{\partial t} + \langle U_i\rangle\frac{\partial\langle u_j u_k\rangle}{\partial x_i} + \frac{\partial\langle u_i u_j u_k\rangle}{\partial x_i} + \langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_i} + \langle u_i u_j\rangle\frac{\partial\langle U_k\rangle}{\partial x_i} \\
&= \nu\nabla^2\langle u_k u_j\rangle - 2\nu\langle\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_i}\rangle - \frac{1}{\rho}\left\langle u_k\frac{\partial p'}{\partial x_j} + u_j\frac{\partial p'}{\partial x_k}\right\rangle, \qquad (10)
\end{aligned}
$$

and this is identical to Eq. (7.178).