

**Turbulent Flows**  
Stephen B. Pope  
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**Solution to Exercise 12.5**

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For a non-symmetric tensor function  $H_{ij}(\mathbf{v})$ , integrating by parts and using the results of Exercise 12.1, we get

$$\begin{aligned}
\int v_k \frac{\partial^2}{\partial v_i \partial v_j} [gH_{ij}(\mathbf{v})] \, d\mathbf{v} &= \int \frac{\partial}{\partial v_i} \left\{ v_k \frac{\partial}{\partial v_j} [gH_{ij}(\mathbf{v})] \right\} \, d\mathbf{v} - \int \frac{\partial}{\partial v_j} [gH_{ij}(\mathbf{v})] \delta_{ik} \, d\mathbf{v} \\
&= \int \frac{\partial}{\partial v_i} \left\{ v_k \frac{\partial}{\partial v_j} [gH_{ij}(\mathbf{v})] \right\} \, d\mathbf{v} - \int \frac{\partial}{\partial v_j} [gH_{kj}(\mathbf{v})] \, d\mathbf{v} \\
&= 0,
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
\int v_k v_\ell \frac{\partial^2}{\partial v_i \partial v_j} [gH_{ij}(\mathbf{v})] \, d\mathbf{v} &= \int \frac{\partial}{\partial v_i} \left( v_k v_\ell \frac{\partial}{\partial v_j} [gH_{ij}(\mathbf{v})] \right) \, d\mathbf{v} \\
&\quad - \int \frac{\partial}{\partial v_j} [gH_{ij}(\mathbf{v})] \frac{\partial v_k v_\ell}{\partial v_i} \, d\mathbf{v} \\
&= - \int v_\ell \frac{\partial}{\partial v_j} [gH_{kj}(\mathbf{v})] \, d\mathbf{v} - \int v_k \frac{\partial}{\partial v_j} [gH_{lj}(\mathbf{v})] \, d\mathbf{v} \\
&= - \int \frac{\partial}{\partial v_j} [g v_\ell H_{kj}(\mathbf{v})] \, d\mathbf{v} + \int g H_{kj}(\mathbf{v}) \delta_{j\ell} \, d\mathbf{v} \\
&\quad - \int \frac{\partial}{\partial v_j} [g v_k H_{lj}(\mathbf{v})] \, d\mathbf{v} + \int g H_{lj}(\mathbf{v}) \delta_{jk} \, d\mathbf{v} \\
&= \int g (H_{k\ell}(\mathbf{v}) + H_{\ell k}(\mathbf{v})) \, d\mathbf{v} \\
&= \langle H_{k\ell}(\mathbf{u}) \rangle + \langle H_{\ell k}(\mathbf{u}) \rangle.
\end{aligned} \tag{2}$$

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