Turbulent Flows
Stephen B. Pope
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Solution to Exercise 12.5

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For a non-symmetric tensor function $H_{ij}(\mathbf{v})$, integrating by parts and using the results of Exercise 12.1, we get

$$\begin{aligned}
\int v_k \frac{\partial^2}{\partial v_i \partial v_j}\left[gH_{ij}(\mathbf{v})\right]\mathrm{d}\mathbf{v} &= \int \frac{\partial}{\partial v_i}\{v_k \frac{\partial}{\partial v_j}\left[gH_{ij}(\mathbf{v})\right]\}\mathrm{d}\mathbf{v} - \int \frac{\partial}{\partial v_j}\left[gH_{ij}(\mathbf{v})\right]\delta_{ik}\mathrm{d}\mathbf{v} \\
&= \int \frac{\partial}{\partial v_i}\{v_k \frac{\partial}{\partial v_j}\left[gH_{ij}(\mathbf{v})\right]\}\mathrm{d}\mathbf{v} - \int \frac{\partial}{\partial v_j}\left[gH_{kj}(\mathbf{v})\right]\mathrm{d}\mathbf{v} \\
&= 0, \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
\int v_k v_\ell \frac{\partial^2}{\partial v_i \partial v_j}\left[gH_{ij}(\mathbf{v})\right]\mathrm{d}\mathbf{v} &= \int \frac{\partial}{\partial v_i}\left(v_k v_\ell \frac{\partial}{\partial v_j}\left[gH_{ij}(\mathbf{v})\right]\right)\mathrm{d}\mathbf{v} \\
&\quad - \int \frac{\partial}{\partial v_j}\left[gH_{ij}(\mathbf{v})\right]\frac{\partial v_k v_\ell}{\partial v_i}\mathrm{d}\mathbf{v} \\
&= -\int v_\ell \frac{\partial}{\partial v_j}\left[gH_{kj}(\mathbf{v})\right]\mathrm{d}\mathbf{v} - \int v_k \frac{\partial}{\partial v_j}\left[gH_{lj}(\mathbf{v})\right]\mathrm{d}\mathbf{v} \\
&= -\int \frac{\partial}{\partial v_j}\left[gv_\ell H_{kj}(\mathbf{v})\right]\mathrm{d}\mathbf{v} + \int gH_{kj}(\mathbf{v})\delta_{j\ell}\mathrm{d}\mathbf{v} \\
&\quad - \int \frac{\partial}{\partial v_j}\left[gv_k H_{\ell j}(\mathbf{v})\right]\mathrm{d}\mathbf{v} + \int gH_{\ell j}(\mathbf{v})\delta_{jk}\mathrm{d}\mathbf{v} \\
&= \int g\left(H_{k\ell}(\mathbf{v}) + H_{\ell k}(\mathbf{v})\right)\mathrm{d}\mathbf{v} \\
&= \langle H_{k\ell}(\mathbf{u})\rangle + \langle H_{\ell k}(\mathbf{u})\rangle. \qquad (2)
\end{aligned}$$