Turbulent Flows
Stephen B. Pope
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Solution to Exercise 12.54

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a): With

$$f'_\phi(\psi;\mathbf{x},t) \equiv \delta(\phi(\mathbf{x},t)-\psi), \tag{1}$$

being the fine-grained PDF of composition, by differentiating with respect to t, we get

$$\begin{aligned}\frac{\partial f'_\phi}{\partial t} &= \frac{\partial \delta(\phi(\mathbf{x},t)-\psi)}{\partial t} \\ &= -\frac{\partial \delta(\phi(\mathbf{x},t)-\psi)}{\partial \psi}\frac{\partial \phi}{\partial t} \\ &= -\frac{\partial}{\partial \psi}\left(f'_\phi \frac{\partial \phi}{\partial t}\right).\end{aligned} \tag{2}$$

b): By differentiating with respect to x_i, we get

$$\begin{aligned}\frac{\partial f'_\phi}{\partial x_i} &= \frac{\partial \delta(\phi(\mathbf{x},t)-\psi)}{\partial x_i} \\ &= -\frac{\partial \delta(\phi(\mathbf{x},t)-\psi)}{\partial \psi}\frac{\partial \phi}{\partial x_i} \\ &= -\frac{\partial}{\partial \psi}\left(f'_\phi \frac{\partial \phi}{\partial x_i}\right).\end{aligned} \tag{3}$$

c): From the definition of the material derivative, we get

$$\frac{\mathrm{D} f'_\phi}{\mathrm{D} t} = \frac{\partial f'_\phi}{\partial t} + U_i \frac{\partial f'_\phi}{\partial x_i}. \tag{4}$$

Substituting Eqs. 2 and 3 into Eq. 4, we get

$$\begin{aligned}
\frac{\mathrm{D}f'_\phi}{\mathrm{D}t} &= -\frac{\partial}{\partial\psi}\left(f'_\phi\frac{\partial\phi}{\partial t}\right) - U_i\frac{\partial}{\partial\psi}\left(f'_\phi\frac{\partial\phi}{\partial x_i}\right) \\
&= -\frac{\partial}{\partial\psi}\left(f'_\phi\frac{\partial\phi}{\partial t} + U_i f'_\phi\frac{\partial\phi}{\partial x_i}\right) \\
&= -\frac{\partial}{\partial\psi}\left(f'_\phi\frac{\mathrm{D}\phi}{\mathrm{D}t}\right). \qquad (5)
\end{aligned}$$

d): Similar to Eq.(12.3) on page 464, we get

$$\begin{aligned}
\langle f'_\phi\mathbf{U}\rangle &= \int\int\delta(\psi'-\psi)\mathbf{V}f_{\mathbf{U}\phi}(\mathbf{V},\psi';\mathbf{x},t)\mathrm{d}\mathbf{V}\mathrm{d}\psi' \\
&= \int\mathbf{V}f_{\mathbf{U}\phi}(\mathbf{V},\psi;\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= f_\phi(\psi;\mathbf{x},t)\int\mathbf{V}f_{\mathbf{U}|\phi}(\mathbf{V}|\psi;\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= f_\phi\left\langle\mathbf{U}(\mathbf{x},t)|\psi\right\rangle = f_\phi\left\langle\langle\mathbf{U}\rangle + \mathbf{u}|\psi\right\rangle \\
&= f_\phi(\langle\mathbf{U}\rangle + \langle\mathbf{u}|\psi\rangle), \qquad (6)
\end{aligned}$$

where the second step follows from the sifting property of the delta function, and $\langle\mathbf{u}|\psi\rangle$ is an abbreviation for $\langle\mathbf{u}(\mathbf{x},t)|\phi(\mathbf{x},t)=\psi\rangle$.

e): From Eq. 5, we have

$$\frac{\mathrm{D}f'_\phi}{\mathrm{D}t} = -\frac{\partial}{\partial\psi}\left(f'_\phi\frac{\mathrm{D}\phi}{\mathrm{D}t}\right). \qquad (7)$$

With Eq. 4 and $\nabla.\mathbf{U}=0$, we get

$$\frac{\partial f'_\phi}{\partial t} + \frac{\partial(U_i f'_\phi)}{\partial x_i} = -\frac{\partial}{\partial\psi}\left(f'_\phi\frac{\mathrm{D}\phi}{\mathrm{D}t}\right). \qquad (8)$$

Taking the mean of Eq. 8 and with Eq. 6, we get

$$\frac{\partial\langle f'_\phi\rangle}{\partial t} + \frac{\partial\langle U_i f'_\phi\rangle}{\partial x_i} = -\frac{\partial}{\partial\psi}\left(\left\langle f'_\phi\frac{\mathrm{D}\phi}{\mathrm{D}t}\right\rangle\right), \qquad (9)$$

i.e.,

$$\begin{aligned}
\frac{\partial f_\phi}{\partial t} &+ \frac{\partial}{\partial x_i}\left(f_\phi[\langle U_i\rangle + \langle u_i|\psi\rangle]\right) = -\frac{\partial}{\partial\psi}\left(f_\phi\left\langle\left.\frac{\mathrm{D}\phi}{\mathrm{D}t}\right|\psi\right\rangle\right) \\
&= -\frac{\partial}{\partial\psi}\{f_\phi[\langle\Gamma\nabla^2\phi|\psi\rangle + S(\psi)]\}, \qquad (10)
\end{aligned}$$

hence verifying Eq.(12.322).

e): Differentiating Eq. 3 with respect to x_i, we get

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{\partial f'_\phi}{\partial x_i}\right) &= -\frac{\partial^2}{\partial x_i \partial \psi}\left(f'_\phi \frac{\partial \phi}{\partial x_i}\right) \\
&= -\frac{\partial}{\partial \psi}\left[\frac{\partial}{\partial x_i}\left(f'_\phi \frac{\partial \phi}{\partial x_i}\right)\right] \\
&= -\frac{\partial}{\partial \psi}\left[f'_\phi \frac{\partial^2 \phi}{\partial x_i \partial x_i} + \frac{\partial \phi}{\partial x_i}\frac{\partial f'_\phi}{\partial x_i}\right] \\
&= -\frac{\partial}{\partial \psi}\left(f'_\phi \frac{\partial^2 \phi}{\partial x_i \partial x_i}\right) + \frac{\partial^2}{\partial \psi^2}\left(f'_\phi \frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\right), \qquad (11)
\end{aligned}$$

i.e.,

$$\nabla^2 f'_\phi = -\frac{\partial}{\partial \psi}\left(f'_\phi \nabla^2 \phi\right) + \frac{\partial^2}{\partial \psi^2}(f'_\phi \nabla\phi \cdot \nabla\phi), \qquad (12)$$

where the last step of Eq. 11 follows from Eq. 3.

f): By taking the mean of Eq. 12, we get

$$\nabla^2 f_\phi = -\frac{\partial}{\partial \psi}\left(f_\phi \langle \nabla^2 \phi | \psi \rangle\right) + \frac{\partial^2}{\partial \psi^2}(f_\phi \langle \nabla\phi \cdot \nabla\phi | \psi \rangle). \qquad (13)$$

Substituting Eq. 13 into Eq. (12.322), we get

$$\begin{aligned}
\frac{\partial f_\phi}{\partial t} &+ \frac{\partial}{\partial x_i}\left(f_\phi[\langle U_i\rangle + \langle u_i|\psi\rangle]\right) = \Gamma \nabla^2 f_\phi \\
&- \frac{\partial^2}{\partial \psi^2}\left(f_\phi \left\langle \Gamma \frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\middle| \psi \right\rangle\right) - \frac{\partial}{\partial \psi}[f_\phi S(\psi)], \qquad (14)
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{\bar{\mathrm{D}} f_\phi}{\bar{\mathrm{D}} t} &= \Gamma \nabla^2 f_\phi - \frac{\partial}{\partial x_i}(f_\phi \langle u_i|\psi\rangle) \\
&- \frac{\partial^2}{\partial \psi^2}\left(f_\phi \left\langle \Gamma \frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i}\middle| \psi \right\rangle\right) - \frac{\partial}{\partial \psi}[f_\phi S(\psi)], \qquad (15)
\end{aligned}$$

hence verifying Eq. (12.323).