Turbulent Flows
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Solution to Exercise 12.56

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Consider a particle method in which the position $\mathbf{X}^*(t)$ and composition $\phi^*(t)$ of a particle evolve by

$$\mathrm{d}\mathbf{X}^* = \mathbf{a}(\mathbf{X}^*, t)\,\mathrm{d}t + b(\mathbf{X}^*, t)\,\mathrm{d}\mathbf{W}, \tag{1}$$

and

$$\mathrm{d}\phi^* = c(\phi^*, \mathbf{X}^*, t)\,\mathrm{d}t, \tag{2}$$

where $\mathbf{a}$, b and c are coefficients and $\mathbf{W}(t)$ is an isotropic Wiener process, we get that $f^*_{\phi\mathbf{X}}(\psi, \mathbf{x}; t)$ evolves by

$$\begin{aligned} \frac{\partial f^*_{\phi\mathbf{X}}}{\partial t} &= -\frac{\partial}{\partial x_i}[f^*_{\phi\mathbf{X}} a_i(\mathbf{x}, t)] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}[f^*_{\phi\mathbf{X}} b^2(\mathbf{x}, t)] \\ &\quad -\frac{\partial}{\partial \psi}[f^*_{\phi\mathbf{X}} c(\psi, \mathbf{x}, t)], \end{aligned} \tag{3}$$

where the first two terms on the right side of Eq. 3 follow from the Fokker-Planck equation (see page 716) and the last term on the right side of Eq. 3 follows from the Liouville equation (see page 716).

a): Integrate this equation over all ψ, we obtain the evolution equation for the particle position density $f^*_{\mathbf{X}}(\mathbf{x}, t)$ as follows

$$\begin{aligned} \int \frac{\partial f^*_{\phi\mathbf{X}}}{\partial t}\mathrm{d}\psi &= -\int \frac{\partial}{\partial x_i}[f^*_{\phi\mathbf{X}} a_i(\mathbf{x}, t)]\mathrm{d}\psi + \int \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}[f^*_{\phi\mathbf{X}} b^2(\mathbf{x}, t)]\mathrm{d}\psi \\ &\quad -\int \frac{\partial}{\partial \psi}[f^*_{\phi\mathbf{X}} c(\psi, \mathbf{x}, t)]\mathrm{d}\psi, \end{aligned} \tag{4}$$

i.e.,

$$\frac{\partial f^*_{\mathbf{X}}}{\partial t} = -\frac{\partial}{\partial x_i}[f^*_{\mathbf{X}} a_i(\mathbf{x}, t)] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}[f^*_{\mathbf{X}} b^2(\mathbf{x}, t)], \tag{5}$$

i.e.,

$$\frac{\partial f^*_{\mathbf{X}}}{\partial t} = \frac{\partial}{\partial x_i}\left\{ f^*_{\mathbf{X}}[-a_i + \frac{\partial}{\partial x_i}(\tfrac{1}{2}b^2)]\right\} + \frac{1}{2}b^2\frac{\partial^2 f^*_{\mathbf{X}}}{\partial x_i \partial x_i}. \tag{6}$$

From Eq. 6, we get

$$\begin{aligned}\frac{\partial f^*_{\mathbf{X}}}{\partial t} &= -a_i(\mathbf{x},t)\frac{\partial f^*_{\mathbf{X}}}{\partial x_i} - f^*_{\mathbf{X}}\frac{\partial a_i(\mathbf{x},t)}{\partial x_i} + \tfrac{1}{2}f^*_{\mathbf{X}}\frac{\partial^2 b^2(\mathbf{x},t)}{\partial x_i \partial x_i} \\ &\quad + \tfrac{1}{2}b^2(\mathbf{x},t)\frac{\partial^2 f^*_{\mathbf{X}}}{\partial x_i \partial x_i} + \frac{\partial f^*_{\mathbf{X}}}{\partial x_i}\frac{\partial b^2(\mathbf{x},t)}{\partial x_i}. \end{aligned} \tag{7}$$

So if the coefficients satisfy

$$\nabla \cdot \mathbf{a} = \tfrac{1}{2}\nabla^2 b^2, \tag{8}$$

from Eq. 7, we get

$$\frac{\partial f^*_{\mathbf{X}}}{\partial t} = -a_i(\mathbf{x},t)\frac{\partial f^*_{\mathbf{X}}}{\partial x_i} + \tfrac{1}{2}b^2(\mathbf{x},t)\frac{\partial^2 f^*_{\mathbf{X}}}{\partial x_i \partial x_i} + \frac{\partial f^*_{\mathbf{X}}}{\partial x_i}\frac{\partial b^2(\mathbf{x},t)}{\partial x_i}. \tag{9}$$

From Eq. 9, we see that if $f^*_{\mathbf{X}}$ is initially uniform, then it will remain uniform.
In short, if $f^*_{\mathbf{X}}$ is initially uniform, then it will remain uniform, provided that the coefficients satisfy

$$\nabla \cdot \mathbf{a} = \tfrac{1}{2}\nabla^2 b^2. \tag{10}$$

b): The PDF of $\phi^*(t)$ conditional on $\mathbf{X}^*(t) = \mathbf{x}$ is defined as

$$f^*_{\phi}(\psi, \mathbf{x}, t) = f^*_{\phi\mathbf{X}}(\psi, \mathbf{x}; t)/f^*_{\mathbf{X}}(\mathbf{x}, t). \tag{11}$$

Taking $f^*_{\mathbf{X}}$ to be uniform, and dividing Eq. 3 by $f^*_{\mathbf{X}}$, we get that f^*_{ϕ} evolves by

$$\begin{aligned}\frac{\partial f^*_{\phi}}{\partial t} &= -\frac{\partial}{\partial x_i}[f^*_{\phi}a_i(\mathbf{x},t)] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}[f^*_{\phi}b^2(\mathbf{x},t)] \\ &\quad - \frac{\partial}{\partial \psi}[f^*_{\phi}c(\psi,\mathbf{x},t)], \end{aligned} \tag{12}$$

If the coefficients are specified by

$$\mathbf{a}(\mathbf{x},t) = \langle \mathbf{U} \rangle + \nabla \Gamma_T, \tag{13}$$

$$b^2(\mathbf{x}, t) = 2\Gamma_T, \tag{14}$$

and

$$c(\psi, \mathbf{x}, t) = -\tfrac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi - \langle \phi^* | \mathbf{x} \rangle) + S(\psi), \tag{15}$$

we get

$$\frac{\bar{\mathrm{D}} f_\phi^*}{\bar{\mathrm{D}} t} = \frac{\partial}{\partial x_i} \left(\Gamma_T \frac{\partial f_\phi^*}{\partial x_i} \right) + \frac{\partial}{\partial \psi} \left\{ f_\phi^* \left[\tfrac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi - \langle \phi^* | \mathbf{x} \rangle) - S(\psi) \right] \right\}, \tag{16}$$

and this is identical to Eq.(12.327) because $f_\phi^* = f_\phi$ and $\langle \phi^* | \mathbf{x} \rangle = \langle \phi \rangle$.