Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 12.57

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We are given that $\phi(\mathbf{x})$ is a statistically homogeneous Gaussian field. Then, at every location, the quantities ϕ, $\partial\phi/\partial x_i$, and $\nabla^2\phi$ are jointly normal random variables, with the means of the gradients being zero.

a): So the covariance of ϕ and $\Gamma\nabla^2\phi$ is

$$\begin{aligned}\langle\phi\Gamma\nabla^2\phi\rangle &= \langle\tfrac{1}{2}\Gamma\nabla^2\phi^2 - \Gamma\nabla\phi\cdot\nabla\phi\rangle \\ &= -\langle\Gamma\nabla\phi\cdot\nabla\phi\rangle \\ &= -\langle\Gamma\nabla\phi'\cdot\nabla\phi'\rangle - \Gamma\nabla\langle\phi\rangle\cdot\nabla\langle\phi\rangle \\ &= -\langle\Gamma\nabla\phi'\cdot\nabla\phi'\rangle \\ &= -\tfrac{1}{2}\varepsilon_\phi, \end{aligned} \tag{1}$$

i.e.,

$$\langle\phi\Gamma\nabla^2\phi\rangle = -\langle\Gamma\nabla\phi\cdot\nabla\phi\rangle = -\tfrac{1}{2}\varepsilon_\phi, \tag{2}$$

where the second and fourth steps in Eq. 1 follow from statistical homogeneity.

b): Using Eq. (3.119) on page 63, we get that the conditional Laplacian is

$$\begin{aligned}\langle\Gamma\nabla^2\phi|\psi\rangle &= \langle\Gamma\nabla^2\phi\rangle + \frac{\langle\Gamma\phi'\nabla^2\phi'\rangle}{\langle\phi'^2\rangle}(\psi - \langle\phi\rangle) \\ &= \frac{\frac{1}{2}\Gamma\nabla^2\langle\phi'^2\rangle}{\langle\phi'^2\rangle}(\psi - \langle\phi\rangle) - \frac{\Gamma\langle\nabla\phi'\cdot\nabla\phi'\rangle}{\langle\phi'^2\rangle}(\psi - \langle\phi\rangle) \\ &= -\frac{1}{2}\frac{\varepsilon_\phi}{\langle\phi'^2\rangle}(\psi - \langle\phi\rangle), \end{aligned} \tag{3}$$

where the second and third steps follow from homogeneity. This is identical to the IEM model, Eq. (12.326) because

$$\frac{\varepsilon_\phi}{\langle\phi'^2\rangle} = C_\phi\frac{\varepsilon}{k}. \tag{4}$$

c):

$$\langle\phi\frac{\partial\phi}{\partial x_i}\rangle = \frac{1}{2}\left\langle\frac{\partial\phi^2}{\partial x_i}\right\rangle = \frac{1}{2}\frac{\partial\langle\phi^2\rangle}{\partial x_i} = 0, \tag{5}$$

where the last step follows from homogeneity. So ϕ and $\partial\phi/\partial x_i$ are uncorrelated and hence independent (because they are jointly normal).

d): So the conditional scalar dissipation (Eq. (12.346)) is

$$\begin{aligned}
\varepsilon_\phi^c(\psi) &\equiv \langle 2\Gamma\nabla\phi'\cdot\nabla\phi'|\phi=\psi\rangle \\
&= \langle 2\Gamma\nabla\phi\cdot\nabla\phi|\phi=\psi\rangle + 2\Gamma\nabla\langle\phi\rangle\cdot\nabla\langle\phi\rangle - 4\Gamma\nabla\langle\phi\rangle\cdot\langle\nabla\phi|\phi=\psi\rangle \\
&= \langle 2\Gamma\nabla\phi\cdot\nabla\phi\rangle + 2\Gamma\nabla\langle\phi\rangle\cdot\nabla\langle\phi\rangle - 4\Gamma\nabla\langle\phi\rangle\cdot\nabla\langle\phi\rangle \\
&= \langle 2\Gamma\nabla\phi\cdot\nabla\phi\rangle \\
&= \langle 2\Gamma\nabla\phi'\cdot\nabla\phi'\rangle + 2\Gamma\nabla\langle\phi\rangle\cdot\nabla\langle\phi\rangle \\
&= \langle 2\Gamma\nabla\phi'\cdot\nabla\phi'\rangle = \varepsilon_\phi, \qquad (6)
\end{aligned}$$

where the third step follows from that ϕ and $\partial\phi/\partial x_i$ are independent and the fourth step follows from homogeneity. So $\varepsilon_\phi^c(\psi)$ is equal to the unconditional scalar dissipation ε_ϕ.

[It is stressed that these results are specific to a Gaussian field, and do not apply in general.]