Turbulent Flows
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Solution to Exercise 12.58

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The evolution equation for the one-point, one-time Eulerian joint PDF of velocity $\mathbf{U}(\mathbf{x}, t)$ and composition $\boldsymbol{\phi}(\mathbf{x}, t)$, $\hat{f}(\mathbf{V}, \boldsymbol{\psi}; x, t)$, is

$$\frac{\partial \hat{f}}{\partial t} + V_i \frac{\partial \hat{f}}{\partial x_i} - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}\frac{\partial \hat{f}}{\partial V_i} + \frac{\partial}{\partial \psi_\alpha}[\hat{f} S_\alpha(\boldsymbol{\psi})] =$$
$$-\frac{\partial}{\partial V_i}\left[\hat{f}\left\langle \nu \nabla^2 U_i - \frac{1}{\rho}\frac{\partial p'}{\partial x_i}\middle| \mathbf{V}, \boldsymbol{\psi}\right\rangle\right] - \frac{\partial}{\partial \psi_\alpha}[\hat{f}\langle \Gamma \nabla^2 \phi_\alpha | \mathbf{V}, \boldsymbol{\psi}\rangle]. \quad (1)$$

Substituting the generalized Langevin model (see Eq. (12.26)) and the IEM model (Eq. (12.353)) for the two unclosed terms on the right-hand side of Eq. 1, we get

$$\frac{\partial \hat{f}}{\partial t} + V_i \frac{\partial \hat{f}}{\partial x_i} - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}\frac{\partial \hat{f}}{\partial V_i} + \frac{\partial}{\partial \psi_\alpha}[\hat{f} S_\alpha(\boldsymbol{\psi})] =$$
$$-\frac{\partial}{\partial V_i}\left[\hat{f} G_{ij}(V_j - \langle U_j \rangle)\right] + \tfrac{1}{2} C_0 \varepsilon \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} + \frac{\partial}{\partial \psi_\alpha}[\tfrac{1}{2} C_\phi \frac{\varepsilon}{k}(\psi_\alpha - \langle \phi_\alpha \rangle)\hat{f}]. \quad (2)$$

a): Multiplying Eq. 2 by V_j and integrating over all the composition space and velocity space, we get

$$\int V_j \frac{\partial \hat{f}}{\partial t} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int V_j V_i \frac{\partial \hat{f}}{\partial x_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \int \frac{1}{\rho} V_j \frac{\partial \langle p \rangle}{\partial x_i}\frac{\partial \hat{f}}{\partial V_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+ \int V_j \frac{\partial}{\partial \psi_\alpha}[\hat{f} S_\alpha(\boldsymbol{\psi})] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} = -\int V_j \frac{\partial}{\partial V_i}\left[\hat{f} G_{ik}(V_k - \langle U_k \rangle)\right] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+ \tfrac{1}{2} C_0 \varepsilon \int V_j \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int V_j \frac{\partial}{\partial \psi_\alpha}[\tfrac{1}{2} C_\phi \frac{\varepsilon}{k}(\psi_\alpha - \langle \phi_\alpha \rangle)\hat{f}] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}. \quad (3)$$

From Eq. 3, we get

$$\frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i U_j \rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i} \int \left(\frac{\partial (V_j \hat{f})}{\partial V_i} - \hat{f}\delta_{ij} \right) \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} =$$

$$-\int \frac{\partial}{\partial V_i}\left[V_j \hat{f} G_{ik}(V_k - \langle U_k\rangle)\right] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int \hat{f} G_{ik}(V_k - \langle U_k\rangle)\delta_{ij} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+\tfrac{1}{2}C_0\varepsilon \int \frac{\partial}{\partial V_i}\left(V_j \frac{\partial \hat{f}}{\partial V_i}\right) \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \tfrac{1}{2}C_0\varepsilon \int \frac{\partial \hat{f}}{\partial V_i}\delta_{ij} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+\int \frac{\partial}{\partial \psi_\alpha}[V_j \tfrac{1}{2}C_\phi \frac{\varepsilon}{k}(\psi_\alpha - \langle \phi_\alpha\rangle)\hat{f}] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \int \frac{\partial}{\partial \psi_\alpha}[V_j \hat{f} S_\alpha(\boldsymbol{\psi})] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} \quad (4)$$

i.e.,

$$\frac{\partial \langle U_j\rangle}{\partial t} + \frac{\partial \langle U_i U_j\rangle}{\partial x_i} + \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_j} =$$
$$\int \hat{f} G_{jk}(V_k - \langle U_k\rangle) \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \tfrac{1}{2}C_0\varepsilon \int \frac{\partial f}{\partial V_j} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}, \quad (5)$$

i.e.,

$$\frac{\partial \langle U_j\rangle}{\partial t} + \frac{\partial \langle U_i U_j\rangle}{\partial x_i} + \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_j} = 0. \quad (6)$$

Form Eq. 6, we get

$$\frac{\partial \langle U_j\rangle}{\partial t} + \frac{\partial \langle U_i\rangle\langle U_j\rangle}{\partial x_i} + \frac{\partial \langle u_i u_j\rangle}{\partial x_i} + \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_j} = 0, \quad (7)$$

i.e.,

$$\frac{\bar{\mathrm{D}}\langle U_j\rangle}{\bar{\mathrm{D}}t} + \frac{\partial \langle u_i u_j\rangle}{\partial x_i} + \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_j} = 0, \quad (8)$$

b): Multiplying Eq. 2 with ψ_β and integrating over all the composition space and velocity space, we get

$$\int \psi_\beta \frac{\partial \hat{f}}{\partial t} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int \psi_\beta V_i \frac{\partial \hat{f}}{\partial x_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \int \frac{1}{\rho}\psi_\beta \frac{\partial \langle p\rangle}{\partial x_i}\frac{\partial \hat{f}}{\partial V_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+\int \psi_\beta \frac{\partial}{\partial \psi_\alpha}[\hat{f} S_\alpha(\boldsymbol{\psi})] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} = -\int \psi_\beta \frac{\partial}{\partial V_i}\left[\hat{f} G_{ik}(V_k - \langle U_k\rangle)\right] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+\tfrac{1}{2}C_0\varepsilon \int \psi_\beta \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int \psi_\beta \frac{\partial}{\partial \psi_\alpha}[\tfrac{1}{2}C_\phi \frac{\varepsilon}{k}(\psi_\alpha - \langle \phi_\alpha\rangle)\hat{f}] \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}. \quad (9)$$

From Eq. 9, we get

$$\frac{\partial \langle \phi_\beta\rangle}{\partial t} + \frac{\partial \langle U_i \phi_\beta\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i}\int \frac{\partial (\psi_\beta \hat{f})}{\partial V_i} \mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$

$$= -\int \frac{\partial}{\partial V_i}\left[\psi_\beta \hat{f} G_{ik}(V_k - \langle U_k\rangle)\right]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \tfrac{1}{2}C_0\varepsilon \int \frac{\partial^2(\psi_\beta \hat{f})}{\partial V_i \partial V_i}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+ \int \frac{\partial}{\partial \psi_\alpha}[\tfrac{1}{2}\psi_\beta C_\phi \frac{\varepsilon}{k}(\psi_\alpha - \langle\phi_\alpha\rangle)\hat{f}]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\int(\psi_\alpha - \langle\phi_\alpha\rangle)\hat{f}\delta_{\alpha\beta}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$- \int \frac{\partial}{\partial \psi_\alpha}[\psi_\beta \hat{f} S_\alpha(\boldsymbol{\psi})]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int \hat{f} S_\alpha(\boldsymbol{\psi})\delta_{\alpha\beta}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}, \tag{10}$$

i.e.,

$$\frac{\partial\langle\phi_\beta\rangle}{\partial t} + \frac{\partial\langle U_i\phi_\beta\rangle}{\partial x_i} = -\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\int(\psi_\alpha - \langle\phi_\alpha\rangle)\hat{f}\delta_{\alpha\beta}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int \hat{f} S_\alpha(\boldsymbol{\psi})\delta_{\alpha\beta}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}, \tag{11}$$

i.e.,

$$\frac{\partial\langle\phi_\beta\rangle}{\partial t} + \frac{\partial\langle U_i\phi_\beta\rangle}{\partial x_i} = \langle S_\beta(\boldsymbol{\phi})\rangle. \tag{12}$$

From Eq. 12, we get

$$\frac{\partial\langle\phi_\beta\rangle}{\partial t} + \frac{\partial(\langle U_i\rangle\langle\phi_\beta\rangle)}{\partial x_i} + \frac{\partial\langle u_i\phi'_\beta\rangle}{\partial x_i} = \langle S_\beta(\boldsymbol{\phi})\rangle, \tag{13}$$

i.e.,

$$\frac{\bar{\mathrm{D}}\langle\phi_\beta\rangle}{\bar{\mathrm{D}}t} + \frac{\partial\langle u_i\phi'_\beta\rangle}{\partial x_i} - \langle S_\beta(\boldsymbol{\phi})\rangle = 0. \tag{14}$$

c): Multiplying Eq. 2 by $V_j\psi_\beta$ and integrating over all the composition space and velocity space, we get

$$\int V_j\psi_\beta\frac{\partial\hat{f}}{\partial t}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int V_j\psi_\beta V_i\frac{\partial\hat{f}}{\partial x_i}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} - \int \frac{1}{\rho}V_j\psi_\beta\frac{\partial\langle p\rangle}{\partial x_i}\frac{\partial\hat{f}}{\partial V_i}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+ \int V_j\psi_\beta\frac{\partial}{\partial\psi_\alpha}[\hat{f}S_\alpha(\boldsymbol{\psi})]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} = -\int V_j\psi_\beta\frac{\partial}{\partial V_i}\left[\hat{f}G_{ik}(V_k - \langle U_k\rangle)\right]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+\tfrac{1}{2}C_0\varepsilon\int V_j\psi_\beta\frac{\partial^2\hat{f}}{\partial V_i\partial V_i}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int V_j\psi_\beta\frac{\partial}{\partial\psi_\alpha}[\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}(\psi_\alpha - \langle\phi_\alpha\rangle)\hat{f}]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}. \tag{15}$$

From Eq. 15, we get

$$\frac{\partial\langle U_j\phi_\beta\rangle}{\partial t} + \frac{\partial\langle U_iU_j\phi_\beta\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i}\int\left(\frac{\partial(V_j\psi_\beta\hat{f})}{\partial V_i} - \hat{f}\psi_\beta\delta_{ij}\right)\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} =$$
$$-\int\frac{\partial}{\partial V_i}\left[V_j\psi_\beta\hat{f}G_{ik}(V_k - \langle U_k\rangle)\right]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi} + \int\hat{f}G_{ik}(V_k - \langle U_k\rangle)\psi_\beta\delta_{ij}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$

$$+\tfrac{1}{2}C_0\varepsilon\int\frac{\partial}{\partial V_i}\left(V_j\psi_\beta\frac{\partial\hat{f}}{\partial V_i}\right)\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}-\tfrac{1}{2}C_0\varepsilon\int\frac{\partial\hat{f}}{\partial V_i}\psi_\beta\delta_{ij}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$+\int\frac{\partial}{\partial\psi_\alpha}[\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}V_j\psi_\beta(\psi_\alpha-\langle\phi_\alpha\rangle)\hat{f}]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\int(\psi_\alpha-\langle\phi_\alpha\rangle)\hat{f}V_j\delta_{\alpha\beta}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}$$
$$-\int\frac{\partial}{\partial\psi_\alpha}[\hat{f}S_\alpha(\boldsymbol{\psi})V_j\psi_\beta]\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi}+\int\hat{f}S_\alpha(\boldsymbol{\psi})V_j\delta_{\alpha\beta}\mathrm{d}\mathbf{V}\mathrm{d}\boldsymbol{\psi},\qquad(16)$$

i.e.,

$$\frac{\partial\langle U_j\phi_\beta\rangle}{\partial t}+\frac{\partial\langle U_iU_j\phi_\beta\rangle}{\partial x_i}+\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}\langle\phi_\beta\rangle=$$
$$G_{jk}\langle u_k\phi_\beta\rangle-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\langle U_j\phi'_\beta\rangle+\langle U_jS_\beta(\boldsymbol{\phi})\rangle,\qquad(17)$$

i.e.,

$$\begin{aligned}\frac{\partial\langle U_j\phi_\beta\rangle}{\partial t}&+\frac{\partial\langle U_iU_j\phi_\beta\rangle}{\partial x_i}+\frac{\langle\phi_\beta\rangle}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}-\langle U_jS_\beta\rangle\\&=G_{j\ell}\langle u_\ell\phi_\beta\rangle-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\langle U_j\phi'_\beta\rangle.\end{aligned}\qquad(18)$$

d): Multiplying Eq. 8 with $\langle\phi_\beta\rangle$, we get

$$\langle\phi_\beta\rangle\frac{\bar{\mathrm{D}}\langle U_j\rangle}{\bar{\mathrm{D}}t}+\langle\phi_\beta\rangle\frac{\partial\langle u_iu_j\rangle}{\partial x_i}+\frac{\langle\phi_\beta\rangle}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}=0,\qquad(19)$$

and multiplying Eq. 14 with $\langle U_j\rangle$, we get

$$\langle U_j\rangle\frac{\bar{\mathrm{D}}\langle\phi_\beta\rangle}{\bar{\mathrm{D}}t}+\langle U_j\rangle\frac{\partial\langle u_i\phi'_\beta\rangle}{\partial x_i}-\langle U_j\rangle\langle S_\beta(\boldsymbol{\phi})\rangle=0.\qquad(20)$$

Adding Eq. 20 to Eq. 19, we get

$$\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\left(\langle U_j\rangle\langle\phi_\beta\rangle\right)+\frac{\partial}{\partial x_i}\Big(\langle\phi_\beta\rangle\langle u_iu_j\rangle+\langle U_j\rangle\langle u_i\phi'_\beta\rangle\Big)-\langle u_iu_j\rangle\frac{\partial\langle\phi_\beta\rangle}{\partial x_i}$$
$$-\langle u_i\phi'_\beta\rangle\frac{\partial\langle U_j\rangle}{\partial x_i}+\frac{\langle\phi_\beta\rangle}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}-\langle U_j\rangle\langle S_\beta(\boldsymbol{\phi})\rangle=0.\qquad(21)$$

From Eq. 18, we get

$$\frac{\partial\langle(\langle U_j\rangle+u_j)(\langle\phi_\beta\rangle+\phi'_\beta)\rangle}{\partial t}+\frac{\partial\langle(\langle U_i\rangle+u_i)(\langle U_j\rangle+u_j)(\langle\phi_\beta\rangle+\phi'_\beta)\rangle}{\partial x_i}$$
$$+\frac{\langle\phi_\beta\rangle}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}-\langle U_j\rangle\langle S_\beta\rangle-\langle u_j S_\beta\rangle=G_{j\ell}\langle u_\ell\phi_\beta\rangle-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\langle U_j\phi'_\beta\rangle, \quad (22)$$

i.e.,

$$\frac{\partial(\langle U_j\rangle\langle\phi_\beta\rangle)}{\partial t}+\frac{\partial\langle u_j\phi'_\beta\rangle}{\partial t}+\langle U_i\rangle\frac{\partial(\langle U_j\rangle\langle\phi_\beta\rangle)}{\partial x_i}+\langle U_i\rangle\frac{\partial\langle u_j\phi'_\beta\rangle}{\partial x_i}$$
$$+\frac{\partial\langle u_iu_j\phi'_\beta\rangle}{\partial x_i}+\frac{\partial(\langle U_j\rangle\langle u_i\phi'_\beta\rangle)}{\partial x_i}+\frac{\partial(\langle u_iu_j\rangle\langle\phi_\beta\rangle)}{\partial x_i}+\frac{\langle\phi_\beta\rangle}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}$$
$$-\langle U_j\rangle\langle S_\beta\rangle-\langle u_jS_\beta\rangle=G_{j\ell}\langle u_\ell\phi'_\beta\rangle-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\langle u_j\phi'_\beta\rangle, \quad (23)$$

Subtracting Eq. 21 from Eq. 23, we get

$$\frac{\partial\langle u_j\phi'_\beta\rangle}{\partial t}+\langle U_i\rangle\frac{\partial\langle u_j\phi'_\beta\rangle}{\partial x_i}+\frac{\partial\langle u_iu_j\phi'_\beta\rangle}{\partial x_i}+\langle u_iu_j\rangle\frac{\partial\langle\phi_\beta\rangle}{\partial x_i}$$
$$+\langle u_i\phi'_\beta\rangle\frac{\partial\langle U_j\rangle}{\partial x_i}-\langle u_jS_\beta\rangle=\left(G_{j\ell}-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\delta_{j\ell}\right)\langle u_\ell\phi'_\beta\rangle, \quad (24)$$

i.e.,

$$\frac{\bar{\mathrm{D}}\langle u_j\phi'_\beta\rangle}{\bar{\mathrm{D}}t}+\frac{\partial\langle u_iu_j\phi'_\beta\rangle}{\partial x_i}+\langle u_iu_j\rangle\frac{\partial\langle\phi_\beta\rangle}{\partial x_i}+\langle u_i\phi'_\beta\rangle\frac{\partial\langle U_j\rangle}{\partial x_i}-\langle u_jS_\beta\rangle$$
$$=\left(G_{j\ell}-\tfrac{1}{2}C_\phi\frac{\varepsilon}{k}\delta_{j\ell}\right)\langle u_\ell\phi'_\beta\rangle. \quad (25)$$