

Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)
Solution to Exercise 12.58

Prepared by: Zhuyin Ren

Date: 05/28/03

The evolution equation for the one-point, one-time Eulerian joint PDF of velocity $\mathbf{U}(\mathbf{x}, t)$ and composition $\phi(\mathbf{x}, t)$, $\hat{f}(\mathbf{V}, \boldsymbol{\psi}; x, t)$, is

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t} + V_i \frac{\partial \hat{f}}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} \frac{\partial \hat{f}}{\partial V_i} + \frac{\partial}{\partial \psi_\alpha} [\hat{f} S_\alpha(\boldsymbol{\psi})] = \\ - \frac{\partial}{\partial V_i} \left[\hat{f} \left\langle \nu \nabla^2 U_i - \frac{1}{\rho} \frac{\partial p'}{\partial x_i} \middle| \mathbf{V}, \boldsymbol{\psi} \right\rangle \right] - \frac{\partial}{\partial \psi_\alpha} [\hat{f} \langle \Gamma \nabla^2 \phi_\alpha | \mathbf{V}, \boldsymbol{\psi} \rangle]. \end{aligned} \quad (1)$$

Substituting the generalized Langevin model (see Eq. (12.26)) and the IEM model (Eq. (12.353)) for the two unclosed terms on the right-hand side of Eq. 1, we get

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t} + V_i \frac{\partial \hat{f}}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} \frac{\partial \hat{f}}{\partial V_i} + \frac{\partial}{\partial \psi_\alpha} [\hat{f} S_\alpha(\boldsymbol{\psi})] = \\ - \frac{\partial}{\partial V_i} [\hat{f} G_{ij} (V_j - \langle U_j \rangle)] + \frac{1}{2} C_0 \varepsilon \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} + \frac{\partial}{\partial \psi_\alpha} [\frac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f}]. \end{aligned} \quad (2)$$

a): Multiplying Eq. 2 by V_j and integrating over all the composition space and velocity space, we get

$$\begin{aligned} \int V_j \frac{\partial \hat{f}}{\partial t} d\mathbf{V} d\boldsymbol{\psi} + \int V_j V_i \frac{\partial \hat{f}}{\partial x_i} d\mathbf{V} d\boldsymbol{\psi} - \int \frac{1}{\rho} V_j \frac{\partial \langle p \rangle}{\partial x_i} \frac{\partial \hat{f}}{\partial V_i} d\mathbf{V} d\boldsymbol{\psi} \\ + \int V_j \frac{\partial}{\partial \psi_\alpha} [\hat{f} S_\alpha(\boldsymbol{\psi})] d\mathbf{V} d\boldsymbol{\psi} = - \int V_j \frac{\partial}{\partial V_i} [\hat{f} G_{ik} (V_k - \langle U_k \rangle)] d\mathbf{V} d\boldsymbol{\psi} \\ + \frac{1}{2} C_0 \varepsilon \int V_j \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} d\mathbf{V} d\boldsymbol{\psi} + \int V_j \frac{\partial}{\partial \psi_\alpha} [\frac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f}] d\mathbf{V} d\boldsymbol{\psi}. \end{aligned} \quad (3)$$

From Eq. 3, we get

$$\frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i U_j \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} \int \left(\frac{\partial (V_j \hat{f})}{\partial V_i} - \hat{f} \delta_{ij} \right) d\mathbf{V} d\boldsymbol{\psi} =$$

$$\begin{aligned}
& - \int \frac{\partial}{\partial V_i} [V_j \hat{f} G_{ik} (V_k - \langle U_k \rangle)] d\mathbf{V} d\boldsymbol{\psi} + \int \hat{f} G_{ik} (V_k - \langle U_k \rangle) \delta_{ij} d\mathbf{V} d\boldsymbol{\psi} \\
& + \frac{1}{2} C_0 \varepsilon \int \frac{\partial}{\partial V_i} \left(V_j \frac{\partial \hat{f}}{\partial V_i} \right) d\mathbf{V} d\boldsymbol{\psi} - \frac{1}{2} C_0 \varepsilon \int \frac{\partial \hat{f}}{\partial V_i} \delta_{ij} d\mathbf{V} d\boldsymbol{\psi} \\
& + \int \frac{\partial}{\partial \psi_\alpha} [V_j \frac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f}] d\mathbf{V} d\boldsymbol{\psi} - \int \frac{\partial}{\partial \psi_\alpha} [V_j \hat{f} S_\alpha(\boldsymbol{\psi})] d\mathbf{V} d\boldsymbol{\psi}
\end{aligned} \tag{4}$$

i.e.,

$$\begin{aligned}
& \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i U_j \rangle}{\partial x_i} + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} = \\
& \int \hat{f} G_{jk} (V_k - \langle U_k \rangle) d\mathbf{V} d\boldsymbol{\psi} - \frac{1}{2} C_0 \varepsilon \int \frac{\partial \hat{f}}{\partial V_j} d\mathbf{V} d\boldsymbol{\psi},
\end{aligned} \tag{5}$$

i.e.,

$$\frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i U_j \rangle}{\partial x_i} + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} = 0. \tag{6}$$

Form Eq. 6, we get

$$\frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i \rangle \langle U_j \rangle}{\partial x_i} + \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} = 0, \tag{7}$$

i.e.,

$$\frac{\bar{D} \langle U_j \rangle}{\bar{D} t} + \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} = 0, \tag{8}$$

b): Multiplying Eq. 2 with ψ_β and integrating over all the composition space and velocity space, we get

$$\begin{aligned}
& \int \psi_\beta \frac{\partial \hat{f}}{\partial t} d\mathbf{V} d\boldsymbol{\psi} + \int \psi_\beta V_i \frac{\partial \hat{f}}{\partial x_i} d\mathbf{V} d\boldsymbol{\psi} - \int \frac{1}{\rho} \psi_\beta \frac{\partial \langle p \rangle}{\partial x_i} \frac{\partial \hat{f}}{\partial V_i} d\mathbf{V} d\boldsymbol{\psi} \\
& + \int \psi_\beta \frac{\partial}{\partial \psi_\alpha} [\hat{f} S_\alpha(\boldsymbol{\psi})] d\mathbf{V} d\boldsymbol{\psi} = - \int \psi_\beta \frac{\partial}{\partial V_i} [\hat{f} G_{ik} (V_k - \langle U_k \rangle)] d\mathbf{V} d\boldsymbol{\psi} \\
& + \frac{1}{2} C_0 \varepsilon \int \psi_\beta \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} d\mathbf{V} d\boldsymbol{\psi} + \int \psi_\beta \frac{\partial}{\partial \psi_\alpha} [\frac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f}] d\mathbf{V} d\boldsymbol{\psi}.
\end{aligned} \tag{9}$$

From Eq. 9, we get

$$\frac{\partial \langle \phi_\beta \rangle}{\partial t} + \frac{\partial \langle U_i \phi_\beta \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} \int \frac{\partial (\psi_\beta \hat{f})}{\partial V_i} d\mathbf{V} d\boldsymbol{\psi}$$

$$\begin{aligned}
&= - \int \frac{\partial}{\partial V_i} [\psi_\beta \hat{f} G_{ik} (V_k - \langle U_k \rangle)] d\mathbf{V} d\boldsymbol{\psi} + \frac{1}{2} C_0 \varepsilon \int \frac{\partial^2 (\psi_\beta \hat{f})}{\partial V_i \partial V_i} d\mathbf{V} d\boldsymbol{\psi} \\
&+ \int \frac{\partial}{\partial \psi_\alpha} [\frac{1}{2} \psi_\beta C_\phi \frac{\varepsilon}{k} (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f}] d\mathbf{V} d\boldsymbol{\psi} - \frac{1}{2} C_\phi \frac{\varepsilon}{k} \int (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f} \delta_{\alpha\beta} d\mathbf{V} d\boldsymbol{\psi} \\
&- \int \frac{\partial}{\partial \psi_\alpha} [\psi_\beta \hat{f} S_\alpha(\boldsymbol{\psi})] d\mathbf{V} d\boldsymbol{\psi} + \int \hat{f} S_\alpha(\boldsymbol{\psi}) \delta_{\alpha\beta} d\mathbf{V} d\boldsymbol{\psi}, \tag{10}
\end{aligned}$$

i.e.,

$$\frac{\partial \langle \phi_\beta \rangle}{\partial t} + \frac{\partial \langle U_i \phi_\beta \rangle}{\partial x_i} = -\frac{1}{2} C_\phi \frac{\varepsilon}{k} \int (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f} \delta_{\alpha\beta} d\mathbf{V} d\boldsymbol{\psi} + \int \hat{f} S_\alpha(\boldsymbol{\psi}) \delta_{\alpha\beta} d\mathbf{V} d\boldsymbol{\psi}, \tag{11}$$

i.e.,

$$\frac{\partial \langle \phi_\beta \rangle}{\partial t} + \frac{\partial \langle U_i \phi_\beta \rangle}{\partial x_i} = \langle S_\beta(\boldsymbol{\phi}) \rangle. \tag{12}$$

From Eq. 12, we get

$$\frac{\partial \langle \phi_\beta \rangle}{\partial t} + \frac{\partial (\langle U_i \rangle \langle \phi_\beta \rangle)}{\partial x_i} + \frac{\partial \langle u_i \phi'_\beta \rangle}{\partial x_i} = \langle S_\beta(\boldsymbol{\phi}) \rangle, \tag{13}$$

i.e.,

$$\frac{\bar{D} \langle \phi_\beta \rangle}{\bar{D} t} + \frac{\partial \langle u_i \phi'_\beta \rangle}{\partial x_i} - \langle S_\beta(\boldsymbol{\phi}) \rangle = 0. \tag{14}$$

c): Multiplying Eq. 2 by $V_j \psi_\beta$ and integrating over all the composition space and velocity space, we get

$$\begin{aligned}
&\int V_j \psi_\beta \frac{\partial \hat{f}}{\partial t} d\mathbf{V} d\boldsymbol{\psi} + \int V_j \psi_\beta V_i \frac{\partial \hat{f}}{\partial x_i} d\mathbf{V} d\boldsymbol{\psi} - \int \frac{1}{\rho} V_j \psi_\beta \frac{\partial \langle p \rangle}{\partial x_i} \frac{\partial \hat{f}}{\partial V_i} d\mathbf{V} d\boldsymbol{\psi} \\
&+ \int V_j \psi_\beta \frac{\partial}{\partial \psi_\alpha} [\hat{f} S_\alpha(\boldsymbol{\psi})] d\mathbf{V} d\boldsymbol{\psi} = - \int V_j \psi_\beta \frac{\partial}{\partial V_i} [\hat{f} G_{ik} (V_k - \langle U_k \rangle)] d\mathbf{V} d\boldsymbol{\psi} \\
&+ \frac{1}{2} C_0 \varepsilon \int V_j \psi_\beta \frac{\partial^2 \hat{f}}{\partial V_i \partial V_i} d\mathbf{V} d\boldsymbol{\psi} + \int V_j \psi_\beta \frac{\partial}{\partial \psi_\alpha} [\frac{1}{2} C_\phi \frac{\varepsilon}{k} (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f}] d\mathbf{V} d\boldsymbol{\psi}. \tag{15}
\end{aligned}$$

From Eq. 15, we get

$$\begin{aligned}
&\frac{\partial \langle U_j \phi_\beta \rangle}{\partial t} + \frac{\partial \langle U_i U_j \phi_\beta \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} \int \left(\frac{\partial (V_j \psi_\beta \hat{f})}{\partial V_i} - \hat{f} \psi_\beta \delta_{ij} \right) d\mathbf{V} d\boldsymbol{\psi} = \\
&- \int \frac{\partial}{\partial V_i} [V_j \psi_\beta \hat{f} G_{ik} (V_k - \langle U_k \rangle)] d\mathbf{V} d\boldsymbol{\psi} + \int \hat{f} G_{ik} (V_k - \langle U_k \rangle) \psi_\beta \delta_{ij} d\mathbf{V} d\boldsymbol{\psi}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}C_0\varepsilon \int \frac{\partial}{\partial V_i} \left(V_j \psi_\beta \frac{\partial \hat{f}}{\partial V_i} \right) d\mathbf{V} d\boldsymbol{\psi} - \frac{1}{2}C_0\varepsilon \int \frac{\partial \hat{f}}{\partial V_i} \psi_\beta \delta_{ij} d\mathbf{V} d\boldsymbol{\psi} \\
& + \int \frac{\partial}{\partial \psi_\alpha} \left[\frac{1}{2}C_\phi \frac{\varepsilon}{k} V_j \psi_\beta (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f} \right] d\mathbf{V} d\boldsymbol{\psi} \\
& - \frac{1}{2}C_\phi \frac{\varepsilon}{k} \int (\psi_\alpha - \langle \phi_\alpha \rangle) \hat{f} V_j \delta_{\alpha\beta} d\mathbf{V} d\boldsymbol{\psi} \\
& - \int \frac{\partial}{\partial \psi_\alpha} [\hat{f} S_\alpha(\boldsymbol{\psi}) V_j \psi_\beta] d\mathbf{V} d\boldsymbol{\psi} + \int \hat{f} S_\alpha(\boldsymbol{\psi}) V_j \delta_{\alpha\beta} d\mathbf{V} d\boldsymbol{\psi}, \tag{16}
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \frac{\partial \langle U_j \phi_\beta \rangle}{\partial t} + \frac{\partial \langle U_i U_j \phi_\beta \rangle}{\partial x_i} + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} \langle \phi_\beta \rangle = \\
& G_{jk} \langle u_k \phi_\beta \rangle - \frac{1}{2}C_\phi \frac{\varepsilon}{k} \langle U_j \phi'_\beta \rangle + \langle U_j S_\beta(\boldsymbol{\phi}) \rangle, \tag{17}
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \frac{\partial \langle U_j \phi_\beta \rangle}{\partial t} + \frac{\partial \langle U_i U_j \phi_\beta \rangle}{\partial x_i} + \frac{\langle \phi_\beta \rangle}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} - \langle U_j S_\beta \rangle \\
& = G_{j\ell} \langle u_\ell \phi_\beta \rangle - \frac{1}{2}C_\phi \frac{\varepsilon}{k} \langle U_j \phi'_\beta \rangle. \tag{18}
\end{aligned}$$

d): Multiplying Eq. 8 with $\langle \phi_\beta \rangle$, we get

$$\langle \phi_\beta \rangle \frac{\bar{D} \langle U_j \rangle}{\bar{D} t} + \langle \phi_\beta \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \frac{\langle \phi_\beta \rangle}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} = 0, \tag{19}$$

and multiplying Eq. 14 with $\langle U_j \rangle$, we get

$$\langle U_j \rangle \frac{\bar{D} \langle \phi_\beta \rangle}{\bar{D} t} + \langle U_j \rangle \frac{\partial \langle u_i \phi'_\beta \rangle}{\partial x_i} - \langle U_j \rangle \langle S_\beta(\boldsymbol{\phi}) \rangle = 0. \tag{20}$$

Adding Eq. 20 to Eq. 19, we get

$$\begin{aligned}
& \frac{\bar{D}}{\bar{D} t} (\langle U_j \rangle \langle \phi_\beta \rangle) + \frac{\partial}{\partial x_i} \left(\langle \phi_\beta \rangle \langle u_i u_j \rangle + \langle U_j \rangle \langle u_i \phi'_\beta \rangle \right) - \langle u_i u_j \rangle \frac{\partial \langle \phi_\beta \rangle}{\partial x_i} \\
& - \langle u_i \phi'_\beta \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\langle \phi_\beta \rangle}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} - \langle U_j \rangle \langle S_\beta(\boldsymbol{\phi}) \rangle = 0. \tag{21}
\end{aligned}$$

From Eq. 18, we get

$$\begin{aligned} & \frac{\partial(\langle U_j \rangle + u_j)(\langle \phi_\beta \rangle + \phi'_\beta)}{\partial t} + \frac{\partial(\langle U_i \rangle + u_i)(\langle U_j \rangle + u_j)(\langle \phi_\beta \rangle + \phi'_\beta)}{\partial x_i} \\ & + \frac{\langle \phi_\beta \rangle}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} - \langle U_j \rangle \langle S_\beta \rangle - \langle u_j S_\beta \rangle = G_{j\ell} \langle u_\ell \phi_\beta \rangle - \frac{1}{2} C_\phi \frac{\varepsilon}{k} \langle U_j \phi'_\beta \rangle, \end{aligned} \quad (22)$$

i.e.,

$$\begin{aligned} & \frac{\partial(\langle U_j \rangle \langle \phi_\beta \rangle)}{\partial t} + \frac{\partial \langle u_j \phi'_\beta \rangle}{\partial t} + \langle U_i \rangle \frac{\partial(\langle U_j \rangle \langle \phi_\beta \rangle)}{\partial x_i} + \langle U_i \rangle \frac{\partial \langle u_j \phi'_\beta \rangle}{\partial x_i} \\ & + \frac{\partial \langle u_i u_j \phi'_\beta \rangle}{\partial x_i} + \frac{\partial(\langle U_j \rangle \langle u_i \phi'_\beta \rangle)}{\partial x_i} + \frac{\partial(\langle u_i u_j \rangle \langle \phi_\beta \rangle)}{\partial x_i} + \frac{\langle \phi_\beta \rangle}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} \\ & - \langle U_j \rangle \langle S_\beta \rangle - \langle u_j S_\beta \rangle = G_{j\ell} \langle u_\ell \phi'_\beta \rangle - \frac{1}{2} C_\phi \frac{\varepsilon}{k} \langle u_j \phi'_\beta \rangle, \end{aligned} \quad (23)$$

Subtracting Eq. 21 from Eq. 23, we get

$$\begin{aligned} & \frac{\partial \langle u_j \phi'_\beta \rangle}{\partial t} + \langle U_i \rangle \frac{\partial \langle u_j \phi'_\beta \rangle}{\partial x_i} + \frac{\partial \langle u_i u_j \phi'_\beta \rangle}{\partial x_i} + \langle u_i u_j \rangle \frac{\partial \langle \phi_\beta \rangle}{\partial x_i} \\ & + \langle u_i \phi'_\beta \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} - \langle u_j S_\beta \rangle = \left(G_{j\ell} - \frac{1}{2} C_\phi \frac{\varepsilon}{k} \delta_{j\ell} \right) \langle u_\ell \phi'_\beta \rangle, \end{aligned} \quad (24)$$

i.e.,

$$\begin{aligned} & \frac{\bar{D} \langle u_j \phi'_\beta \rangle}{\bar{D} t} + \frac{\partial \langle u_i u_j \phi'_\beta \rangle}{\partial x_i} + \langle u_i u_j \rangle \frac{\partial \langle \phi_\beta \rangle}{\partial x_i} + \langle u_i \phi'_\beta \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} - \langle u_j S_\beta \rangle \\ & = \left(G_{j\ell} - \frac{1}{2} C_\phi \frac{\varepsilon}{k} \delta_{j\ell} \right) \langle u_\ell \phi'_\beta \rangle. \end{aligned} \quad (25)$$

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