Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 2.8

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a) From Eq. (A.61) we obtain

$$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \mathbf{U}) = \varepsilon_{ijk} \frac{\partial^2 U_k}{\partial x_i \partial x_j} = 0. \tag{1}$$

The last step follows since $\frac{\partial^2 U_k}{\partial x_i \partial x_j}$ is symmetric with respect to i and j. For a symmetric quantity $S_{ij} = S_{ji}$ we have

$$\varepsilon_{ijk} S_{ij} = \varepsilon_{jik} S_{ji} = -\varepsilon_{ijk} S_{ij} = 0. \tag{2}$$

b)

$$\nabla \times \nabla \phi = \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} = 0, \tag{3}$$

for the same reasons as part (a).

c)

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{U}) &= \underline{e}_i \left(\varepsilon_{ijk} \frac{\partial}{\partial x_j} \right) \left(\varepsilon_{k\ell m} \frac{\partial}{\partial x_\ell} \right) U_m \\ &= \underline{e}_i \varepsilon_{ijk} \varepsilon_{k\ell m} \left(\frac{\partial^2 U_m}{\partial x_j \partial x_\ell} \right). \end{aligned} \tag{4}$$

where (see appendix A, exercise A.11)

$$\varepsilon_{ijk} \varepsilon_{k\ell m} = \varepsilon_{ijk} \varepsilon_{\ell mk} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell} \tag{5}$$

therefore,

$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{U}) &= \underline{e}_i \left[\delta_{i\ell}\delta_{jm} \frac{\partial^2 U_m}{\partial x_j \partial x_\ell} - \delta_{im}\delta_{j\ell} \frac{\partial^2 U_m}{\partial x_j \partial x_\ell} \right] \\
&= \underline{e}_i \left[\frac{\partial^2 U_j}{\partial x_j \partial x_i} - \frac{\partial^2 U_i}{\partial x_j \partial x_j} \right] \\
&= \nabla(\nabla \cdot \mathbf{U}) - \nabla^2 \mathbf{U} \qquad (6)
\end{aligned}
$$

d)

$$
\begin{aligned}
\mathbf{U} \times \underline{\omega} &= \mathbf{U} \times (\nabla \times \mathbf{U}) \\
&= \underline{e}_i \varepsilon_{ijk} \varepsilon_{k\ell m} u_j \frac{\partial u_m}{\partial x_\ell} \\
&= \underline{e}_i (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) u_j \frac{\partial u_m}{\partial x_\ell} \\
&= \underline{e}_i \left(u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} \right) \\
&= \underline{e}_i \left(\frac{1}{2} \frac{\partial (u_j u_j)}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} \right) \\
&= \frac{1}{2} \nabla (\mathbf{U} \cdot \mathbf{U}) - \mathbf{U} \cdot \nabla \mathbf{U}. \qquad (7)
\end{aligned}
$$

f) Are the expressions in Eqs. (2.64) and (2.65) tensors?

Yes. It is clear that the quantities on the right-hand sides of Eqs. (2.64) and (2.65) are first-order tensors. The quantities on the left-hand sides contain the alternating tensor twice, and hence transform as tensors under reflections of the coordinate axes, see Eq. (5).