Turbulent Flows
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Solution to Exercise 2.9

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The Navier-Stokes equations (Eq.2.35) are

$$\frac{D\mathbf{U}}{Dt} = \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{U}. \tag{1}$$

Hence, using Eq.2.65, the left-hand side can be re-expressed as

$$\frac{D\mathbf{U}}{Dt} = \frac{\partial \mathbf{U}}{\partial t} + \nabla\left(\tfrac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) - \mathbf{U}\times\boldsymbol{\omega}, \tag{2}$$

which yields the *Stokes form* of the Navier-Stokes equations when plugged into eq. (1).

$$\frac{\partial \mathbf{U}}{\partial t} - \mathbf{U}\times\boldsymbol{\omega} + \nabla\left(\tfrac{1}{2}\mathbf{U}\cdot\mathbf{U} + \frac{p}{\rho}\right) = \nu\nabla^2\mathbf{U}. \tag{3}$$

Assuming steady, inviscid flow, eq. 3 can be simplified to

$$\nabla H = \mathbf{U}\times\boldsymbol{\omega}, \tag{4}$$

where H is the Bernoulli integral defined in eq. (2.67).

a) Applying the operator $\mathbf{U}\cdot$ to eq. 4 yields

$$\mathbf{U}\cdot\nabla H = \mathbf{U}\cdot\left(\mathbf{U}\times\boldsymbol{\omega}\right). \tag{5}$$

The pseudovector $\mathbf{U}\times\boldsymbol{\omega}$ is perpendicular to both $\mathbf{U}$ and $\boldsymbol{\omega}$ due to the properties of the cross product, and thus the scalar product with $\mathbf{U}$ is equal to zero.

$$\mathbf{U}\cdot\nabla H = 0. \tag{6}$$

Eq. 6 implies that the gradient of the Bernoulli integral is perpendicular to $\mathbf{U}$, and hence its value does not change in the direction of $\mathbf{U}$, i.e. along streamlines.

b) Following the same arguments as in a), applying the operator $\boldsymbol{\omega}\cdot$ to eq. 4 yields

$$\boldsymbol{\omega} \cdot \nabla H = 0, \tag{7}$$

and hence the value of H does not change along vortex lines either.

c) If the flow is irrotational, $\boldsymbol{\omega} = 0$ and eq. 4 reduces to

$$\nabla H = 0. \tag{8}$$

Since its gradient is zero everywhere in the flow, the Bernoulli integral must be a constant.