

Turbulent Flows
Stephen B. Pope
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Solution to Exercise 2.9

Prepared by: Michael W. T. Stumpf

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The Navier-Stokes equations (Eq.2.35) are

$$\frac{D\mathbf{U}}{Dt} = \frac{\partial\mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{U}. \quad (1)$$

Hence, using Eq.2.65, the left-hand side can be re-expressed as

$$\frac{D\mathbf{U}}{Dt} = \frac{\partial\mathbf{U}}{\partial t} + \nabla \left(\frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) - \mathbf{U} \times \boldsymbol{\omega}, \quad (2)$$

which yields the *Stokes form* of the Navier-Stokes equations when plugged into eq. (1).

$$\frac{\partial\mathbf{U}}{\partial t} - \mathbf{U} \times \boldsymbol{\omega} + \nabla \left(\frac{1}{2} \mathbf{U} \cdot \mathbf{U} + \frac{p}{\rho} \right) = \nu \nabla^2 \mathbf{U}. \quad (3)$$

Assuming steady, inviscid flow, eq. 3 can be simplified to

$$\nabla H = \mathbf{U} \times \boldsymbol{\omega}, \quad (4)$$

where H is the Bernoulli integral defined in eq. (2.67).

a) Applying the operator $\mathbf{U} \cdot$ to eq. 4 yields

$$\mathbf{U} \cdot \nabla H = \mathbf{U} \cdot (\mathbf{U} \times \boldsymbol{\omega}). \quad (5)$$

The pseudovector $\mathbf{U} \times \boldsymbol{\omega}$ is perpendicular to both \mathbf{U} and $\boldsymbol{\omega}$ due to the properties of the cross product, and thus the scalar product with \mathbf{U} is equal to zero.

$$\mathbf{U} \cdot \nabla H = 0. \quad (6)$$

Eq. 6 implies that the gradient of the Bernoulli integral is perpendicular to \mathbf{U} , and hence its value does not change in the direction of \mathbf{U} , i.e. along streamlines.

- b) Following the same arguments as in a), applying the operator $\boldsymbol{\omega} \cdot$ to eq. 4 yields

$$\boldsymbol{\omega} \cdot \nabla H = 0, \quad (7)$$

and hence the value of H does not change along vortex lines either.

- c) If the flow is irrotational, $\boldsymbol{\omega} = 0$ and eq. 4 reduces to

$$\nabla H = 0. \quad (8)$$

Since its gradient is zero everywhere in the flow, the Bernoulli integral must be a constant.

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