Exercise 3.25 Solution

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The mean and the variance of the random vector

$$\widehat{\mathbf{u}} \equiv \mathbf{C}^{-1/2}(\mathbf{U} - \mu) \tag{1}$$

are

$$\langle \widehat{\mathbf{u}} \rangle = \mathbf{C}^{-1/2}(\langle \mathbf{U} \rangle - \mu) = 0 \tag{2}$$

and

$$\begin{aligned} \langle \widehat{\mathbf{u}}\widehat{\mathbf{u}}^T \rangle =& \mathbf{C}^{-1/2} \langle \mathbf{u}\mathbf{u}^{\mathbf{T}} \rangle \mathbf{C}^{-1/2} \\ =& \mathbf{C}^{-1/2}\mathbf{C}\mathbf{C}^{-1/2} = \mathbf{I} \end{aligned} \tag{3}$$

where $\mathbf{I}$ is the identity. This establishes that the components of $\widehat{\mathbf{u}}$ are uncorrelated standardised random variables. Equation (1) defines $\widehat{\mathbf{u}}$ as a linear transformation of jointly normal random variables, and hence $\widehat{\mathbf{u}}$ is itself jointly normal. (See Appendix I). Since the components of $\widehat{\mathbf{u}}$ are uncorrelated and $\widehat{\mathbf{u}}$ is joint normal, it follows that the components are independent. Hence their joint PDF is the product of their marginals

$$\begin{aligned} \widehat{f}(\widehat{\mathbf{v}}) =& \prod_{i=1}^{D} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\widehat{V}_{(i)}\widehat{V}_{(i)}) \\ =& (\frac{1}{2\pi})^{D/2} \exp(-\frac{1}{2}\widehat{V}_i\widehat{V}_i) \\ =& (\frac{1}{2\pi})^{D/2} \exp(-\frac{1}{2}\widehat{\mathbf{V}}^T\widehat{\mathbf{V}}) \end{aligned} \tag{4}$$

The result is also readily obtained via characteristic functions, see Appendix I.