Exercise 3.28 Solution

S B Pope

Febuary 3, 2002

We have

$$u^2 \frac{d^3u}{dt^3} = \frac{d}{dt}(u^2 \frac{d^2u}{dt^2}) - \frac{du^2}{dt}\frac{d^2u}{dt^2} \tag{1}$$

For a stationary process, the mean of the first term on the right hand side is zero. Hence we obtain

$$\begin{aligned} \langle u^2 \frac{d^3u}{dt^3}\rangle &= - \langle \frac{du^2}{dt}\frac{d^2u}{dt^2}\rangle \\ &= - \langle 2u \frac{du}{dt}\frac{d^2}{dt^2}\rangle \\ &= - 2\langle u\dot{u}\ddot{u}\rangle \end{aligned} \tag{2}$$

Now

$$-2u\frac{du}{dt}\frac{d^2u}{dt^2} = \frac{d}{dt}[(-2u\frac{du}{dt})\frac{du}{dt}] + \frac{du}{dt}\frac{d}{dt}(2u\frac{du}{dt}) \tag{3}$$

so that

$$\begin{aligned} -2\langle u\dot{u}\ddot{u}\rangle &= \langle \frac{du}{dt}\frac{d}{dt}(2u\frac{du}{dt})\rangle \\ &= 2\langle \dot{u}(\dot{u}^2 + u\ddot{u})\rangle \\ &= 2\langle \dot{u}^3\rangle + 2\langle u\dot{u}\ddot{u}\rangle \end{aligned} \tag{4}$$

This equation yields

$$-4\langle u\dot{u}\ddot{u}\rangle = 2\langle \dot{u}^3\rangle \tag{5}$$

which, substituted into Eq. (2) leads to

$$\begin{aligned}\langle u^2 \frac{d^3u}{dt^3} \rangle =& - 2\langle u\dot{u}\ddot{u}\rangle = 2\langle \dot{u}^3\rangle + 2\langle u\dot{u}\ddot{u}\rangle \\ =&\langle \dot{u}^3\rangle \end{aligned} \tag{6}$$