Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 3.30

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Based on chain rules, we have

$$\frac{\mathrm{d}u(t+s)}{\mathrm{d}(t+s)} = \frac{\partial u(t+s)}{\partial t} = \frac{\partial u(t+s)}{\partial s}. \tag{1}$$

$$\frac{\mathrm{d}^2u(t+s)}{\mathrm{d}^2(t+s)} = \frac{\partial^2 u(t+s)}{\partial t^2} = \frac{\partial^2 u(t+s)}{\partial s^2}. \tag{2}$$

So we have

$$\begin{aligned}
B(s) &= \langle \dot{u}(t)\dot{u}(t+s)\rangle \\
&= \langle \frac{\mathrm{d}u(t)}{\mathrm{d}t}\frac{\mathrm{d}u(t+s)}{\mathrm{d}(t+s)}\rangle \\
&= \langle \frac{\partial}{\partial t}\left(u(t)\frac{\partial u(t+s)}{\partial t}\right) - u(t)\frac{\partial^2 u(t+s)}{\partial t^2}\rangle \\
&= \frac{\partial}{\partial t}\langle u(t)\frac{\partial u(t+s)}{\partial t}\rangle - \langle u(t)\frac{\partial^2 u(t+s)}{\partial t^2}\rangle
\end{aligned} \tag{3}$$

Since $u(t)$ is a statistically stationary process

$$\frac{\partial}{\partial t}\langle u(t)\frac{\partial u(t+s)}{\partial t}\rangle = 0. \tag{4}$$

So it can be shown that

$$
\begin{aligned}
B(s) &= -\langle u(t)\frac{\partial^2 u(t+s)}{\partial s^2}\rangle \\
&= -\langle \frac{\partial^2}{\partial s^2}\left(u(t)u(t+s)\right)\rangle \\
&= -\frac{\partial^2}{\partial s^2}\langle (u(t)u(t+s))\rangle \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\langle (u(t)u(t+s))\rangle \\
&= -\frac{\mathrm{d}^2 R(s)}{\mathrm{d}s^2} \qquad (5)
\end{aligned}
$$