

Turbulent Flows
Stephen B. Pope
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Solution to Exercise 3.5

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a) The normalization is given by Eq. (3.16), i.e.,

$$\int_{-\infty}^{\infty} f(V) dV = 1. \quad (1)$$

For the exponential distribution of $f(V)$ is given by Eq. (3.40)

$$f(V) = \begin{cases} \frac{1}{\lambda} \exp\left(-\frac{V}{\lambda}\right) & \text{for } V \geq 0 \\ 0 & \text{for } V < 0. \end{cases} \quad (2)$$

Substituting Eq. (2) in Eq. (1) we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f(V) dV &= \int_0^{\infty} \frac{1}{\lambda} \exp\left(-\frac{V}{\lambda}\right) dV \\ &= \int_{-\infty}^0 \exp(t) dt, \quad t = -\frac{V}{\lambda} \\ &= 1. \end{aligned}$$

Hence the exponential distribution satisfies the normalization condition.

b) In general, the mean of U is given by Eq. (3.19)

$$\langle U \rangle = \int_{-\infty}^{\infty} V f(V) dV. \quad (3)$$

With $f(V)$ being the exponential distribution, (given by Eq. 3.40), Eq. (3) becomes

$$\begin{aligned}
\langle U \rangle &= \int_0^\infty \frac{V}{\lambda} \exp\left(-\frac{V}{\lambda}\right) dV \\
&= -\lambda \int_{-\infty}^0 t \exp(t) dt, \quad t = -\frac{V}{\lambda} \\
&= -\lambda \int_{-\infty}^0 \left[\frac{d}{dt} \{t \exp(t)\} - \exp(t) \right] dt \\
&= \lambda \int_{-\infty}^0 \exp(t) dt \\
&= \lambda.
\end{aligned} \tag{4}$$

c) Using Eq. (3.20) with $Q(U) = U^n$ we obtain

$$\begin{aligned}
\langle U^n \rangle &= \int_0^\infty \frac{V^n}{\lambda} \exp\left(-\frac{V}{\lambda}\right) dV \\
&= \int_0^\infty \left[nV^{(n-1)} \exp\left(-\frac{V}{\lambda}\right) - \frac{d}{dV} \left\{ V^n \exp\left(-\frac{V}{\lambda}\right) \right\} \right] dV \\
&= n \int_0^\infty V^{(n-1)} \exp\left(-\frac{V}{\lambda}\right) dV \\
\langle U^n \rangle &= n\lambda \langle U^{(n-1)} \rangle
\end{aligned} \tag{5}$$

d) Eq. (3.14) states,

$$f(V) \equiv \frac{dF(V)}{dV} \tag{6}$$

Using Eq. (2) in Eq. (6) and integrating with respect to V we get ,

$$F(V) = \begin{cases} 0 & \text{for } V < 0 \\ 1 - \exp\left(-\frac{V}{\lambda}\right) & \text{for } V \geq 0. \end{cases}$$

Note for $V = 0$, $F(V) = 0$, so we can rewrite $F(V)$ as

$$F(V) = \begin{cases} 0 & \text{for } V \leq 0 \\ 1 - \exp\left(-\frac{V}{\lambda}\right) & \text{for } V > 0. \end{cases} \quad (7)$$

e) Using Eq (3.13) and setting $V_a = a\lambda$ and $V_b = \infty$ we obtain,

$$\begin{aligned} P\{a\lambda \leq U < \infty\} &= F(\infty) - F(a\lambda) \\ P\{U \geq a\lambda\} &= 1 - F(a\lambda) \end{aligned}$$

Using Eq. (7) and $a \geq 0$ we get

$$\begin{aligned} P\{U \geq a\lambda\} &= 1 - \left[1 - \exp\left(-a\frac{\lambda}{\lambda}\right)\right] \\ &= \exp(-a). \end{aligned} \quad (8)$$

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