Turbulent Flows
Stephen B. Pope
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Solution to Exercise 3.5

Prepared by: Sathyanarayana Ayyalasomayajula *Date:* 02/09/03

a) The normalization is given by Eq. (3.16), i.e.,

$$\int_{-\infty}^{\infty} f(V)\,\mathrm{d}V = 1. \tag{1}$$

For the exponential distribution of $f(V)$ is given by Eq. (3.40)

$$f(V) = \begin{cases} \frac{1}{\lambda}\exp\left(-\frac{V}{\lambda}\right) & \text{for V} \geq 0 \\ 0 & \text{for V} < 0. \end{cases} \tag{2}$$

Substituting Eq. (2) in Eq. (1) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(V)\,\mathrm{d}V &= \int_{0}^{\infty} \frac{1}{\lambda}\exp\left(-\frac{V}{\lambda}\right)\mathrm{d}V \\ &= \int_{-\infty}^{0} \exp(t)\,\mathrm{d}t,\ \ t = -\frac{V}{\lambda} \\ &= 1.\end{aligned}$$

Hence the exponential distribution satisfies the normalization condition.

b) In general, the mean of U is given by Eq. (3.19)

$$\langle U\rangle = \int_{-\infty}^{\infty} V f(V)\,\mathrm{d}V. \tag{3}$$

With $f(V)$ being the exponential distribution, (given by Eq. 3.40), Eq. (3) becomes

$$\begin{aligned}
\langle U \rangle &= \int_0^\infty \frac{V}{\lambda} \exp\left(-\frac{V}{\lambda}\right) \mathrm{d}V \\
&= -\lambda \int_{-\infty}^0 t \exp(t)\,\mathrm{d}t, \;\; t = -\frac{V}{\lambda} \\
&= -\lambda \int_{-\infty}^0 \left[\frac{\mathrm{d}}{\mathrm{d}t} \{t \exp(t)\} - \exp(t) \right] \mathrm{d}t \\
&= \lambda \int_{-\infty}^0 exp(t)\,\mathrm{d}t \\
&= \lambda. \qquad (4)
\end{aligned}$$

c) Using Eq. (3.20) with $Q(U) = U^n$ we obtain

$$\begin{aligned}
\langle U^n \rangle &= \int_0^\infty \frac{V^n}{\lambda} exp\left(-\frac{V}{\lambda}\right) \mathrm{d}V \\
&= \int_0^\infty \left[nV^{(n-1)} \exp\left(-\frac{V}{\lambda}\right) - \frac{\mathrm{d}}{\mathrm{d}V} \left\{ V^n \exp\left(-\frac{V}{\lambda}\right) \right\} \right] \mathrm{d}V \\
&= n \int_0^\infty V^{(n-1)} exp\left(-\frac{V}{\lambda}\right) \mathrm{d}V \\
\langle U^n \rangle &= n\lambda \langle U^{(n-1)} \rangle \qquad (5)
\end{aligned}$$

d) Eq. (3.14) states,

$$f(V) \equiv \frac{\mathrm{d}F(V)}{\mathrm{d}V} \qquad (6)$$

Using Eq. (2) in Eq. (6) and integrating with respect to V we get ,

$$F(V) = \begin{cases} 0 & \text{for V} < 0 \\ 1 - \exp\left(-\frac{V}{\lambda}\right) & \text{for V} \geq 0. \end{cases}$$

Note for $V = 0, F(V) = 0$, so we can rewrite $F(V)$ as

$$F(V) = \begin{cases} 0 & \text{for V} \leq 0 \\ 1 - \exp\left(-\frac{V}{\lambda}\right) & \text{for V} > 0. \end{cases} \tag{7}$$

e) Using Eq (3.13) and setting $V_a = a\lambda$ and $V_b = \infty$ we obtain,

$$\begin{aligned} P\{a\lambda \leq U < \infty\} &= F(\infty) - F(a\lambda) \\ P\{U \geq a\lambda\} &= 1 - F(a\lambda) \end{aligned}$$

Using Eq. (7) and $a \geq 0$ we get

$$\begin{aligned} P\{U \geq a\lambda\} &= 1 - \left[1 - \exp\left(-a\frac{\lambda}{\lambda}\right)\right] \\ &= \exp(-a)\,. \end{aligned} \tag{8}$$