

Turbulent Flows
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Solution to Exercise 3.4

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By definition $Y = e^U$ (Eq. 3.49) where U is normally distributed with mean μ and variance σ^2 . Now

$$\begin{aligned}\langle Y^n \rangle &= \langle e^{nU} \rangle \\ &= \int_{-\infty}^{\infty} e^{nV} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}[V - \mu]^2 / \sigma^2\right) dV \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}[V - \mu]^2 / \sigma^2 + nV\right) dV\end{aligned}\quad (1)$$

Now the argument of the exponential is:

$$\begin{aligned}-\frac{1}{2\sigma^2}[V - \mu]^2 + nV &= \frac{1}{2\sigma^2}[-V^2 + 2V\mu - \mu^2 + 2\sigma^2Vn] \\ &= \frac{1}{2\sigma^2}[-V^2 + 2V(\mu + \sigma^2n) - \mu^2 - (\mu + \sigma^2n)^2 + (\mu + \sigma^2n)^2] \\ &= \frac{1}{2\sigma^2}[-(V - [\mu + \sigma^2n])^2 - \mu^2 + (\mu + \sigma^2n)^2] \\ &= -\frac{1}{2\sigma^2}(V - [\mu + \sigma^2n])^2 + (n\mu + \frac{1}{2}\sigma^2n^2)\end{aligned}\quad (2)$$

Thus

$$\begin{aligned}\langle Y^n \rangle &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(V - [\mu + \sigma^2n])^2 / \sigma^2\right) dV \times \exp\left(n\mu + \frac{1}{2}\sigma^2n^2\right) \\ &= \exp\left(n\mu + \frac{1}{2}\sigma^2n^2\right)\end{aligned}\quad (3)$$

For the case $\mu = -\frac{1}{2}\sigma^2$, for $n=1$ we obtain

$$\langle Y \rangle = \exp\left(\mu + \frac{1}{2}\sigma^2\right) = e^0 = 1\quad (4)$$

and for $n=2$

$$\langle Y^2 \rangle = \exp(2\mu + 2\sigma^2) = \exp(\sigma^2) \quad (5)$$

Thus

$$\text{var}(Y) = \langle Y^2 \rangle - \langle Y \rangle^2 = \langle Y \rangle^2 (\exp(\sigma^2) - 1) \quad (6)$$

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