Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 3.4

Prepared by: Stephen B. Pope   *Date:* 02/27/03

By defination $Y = e^U$ (Eq. 3.49) where U is normally distributed with mean μ and variance σ^2. Now

$$\begin{aligned} \langle Y^n \rangle &= \langle e^{nU} \rangle \\ &= \int_{-\infty}^{\infty} e^{nV} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\tfrac{1}{2}\left[V-\mu\right]^2/\sigma^2\right) \mathrm{d}V \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\tfrac{1}{2}\left[V-\mu\right]^2/\sigma^2 + nV\right) \mathrm{d}V \qquad (1) \end{aligned}$$

Now the argument of the exponential is:

$$\begin{aligned} -\frac{1}{2\sigma^2}\left[V-\mu\right]^2 + nV &= \frac{1}{2\sigma^2}\left[-V^2 + 2V\mu - \mu^2 + 2\sigma^2 V n\right] \\ &= \frac{1}{2\sigma^2}\left[-V^2 + 2V(\mu+\sigma^2 n) - \mu^2 - (\mu+\sigma^2 n)^2 + (\mu+\sigma^2 n)^2\right] \\ &= \frac{1}{2\sigma^2}\left[-(V-[\mu+\sigma^2 n])^2 - \mu^2 + (\mu+\sigma^2 n)^2\right] \\ &= -\frac{1}{2\sigma^2}\left(V - \left[\mu+\sigma^2 n\right]\right)^2 + \left(n\mu + \tfrac{1}{2}\sigma^2 n^2\right) \qquad (2) \end{aligned}$$

Thus

$$\begin{aligned} \langle Y^n \rangle &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\tfrac{1}{2}(V-[\mu+\sigma^2 n])^2/\sigma^2\right) \mathrm{d}V \times exp\left(n\mu + \tfrac{1}{2}\sigma^2 n^2\right) \\ &= exp\left(n\mu + \tfrac{1}{2}\sigma^2 n^2\right) \qquad (3) \end{aligned}$$

For the case $\mu = -\frac{1}{2}\sigma^2$,for n=1 we obtain

$$\langle Y \rangle = exp\left(\mu + \tfrac{1}{2}\sigma^2\right) = e^0 = 1 \qquad (4)$$

and for n=2

$$\langle Y^2 \rangle = exp\left(2\mu + 2\sigma^2\right) = exp\left(\sigma^2\right) \tag{5}$$

Thus

$$\mathrm{var}(Y) = \langle Y^2 \rangle - \langle Y \rangle^2 = \langle Y \rangle^2 \left(exp(\sigma^2) - 1\right) \tag{6}$$