Solution to Exercise 5.13

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The Cauchy-Schwarz inequality states

$$2ab \leq a^2 + b^2 \, , \{a,b\} \in \mathbb{R}_0^+ \tag{1}$$

To apply the upper inequality, one must ensure that the fluctuating velocities are non-negative. We take the absolute value:

$$
\begin{aligned}
2|u||v| &\leq |u|^2 + |v|^2 \\
\overset{u,v \in \mathbb{R}}{\Longleftrightarrow} |uv| &\leq \left(\frac{u^2}{2} + \frac{v^2}{2} \right) \\
\overset{\langle \cdot \rangle}{\Longleftrightarrow} \langle |uv| \rangle &\leq \left\langle \frac{1}{2}\left(u^2 + v^2 \right) \right\rangle \\
\Leftrightarrow |\langle uv \rangle| &\leq \frac{1}{2}\left(\langle u^2 \rangle + \langle v^2 \rangle \right) \leq \frac{1}{2}\left(\langle u^2 \rangle + \langle v^2 \rangle + \langle w^2 \rangle \right) = k \tag{2}
\end{aligned}
$$