Turbulent Flows
Stephen B. Pope
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Solution to Exercise 5.14

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We first show the relation between the material derivative of energy and the advected material derivative of velocity:

$$\begin{aligned}
\boldsymbol{U} \cdot \frac{D\boldsymbol{U}}{Dt} &= \boldsymbol{U} \cdot \left(\frac{\partial \boldsymbol{U}}{\partial t} + \big(\boldsymbol{U} \cdot \nabla\big)\boldsymbol{U} \right) && (1)\\
&= \boldsymbol{U} \cdot \frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \Big(\big(\boldsymbol{U} \cdot \nabla\big)\boldsymbol{U} \Big)\\
&= U_i \frac{\partial U_i}{\partial t} + U_i \Big(\big(U_j \nabla_j\big) U_i \Big)\\
&= \frac{1}{2} \frac{\partial}{\partial t} \big(U_i U_i\big) + U_j U_i \nabla_j U_i\\
&= \frac{1}{2} \frac{\partial}{\partial t} \big(U_i U_i\big) + \frac{1}{2} U_j \nabla_j \big(U_i U_i\big)\\
&= \frac{1}{2} \frac{\partial}{\partial t} \big(\boldsymbol{U} \cdot \boldsymbol{U}\big) + \frac{1}{2} \boldsymbol{U} \cdot \nabla \big(\boldsymbol{U} \cdot \boldsymbol{U}\big)\\
&= \frac{\partial}{\partial t} \left(\frac{1}{2} \boldsymbol{U} \cdot \boldsymbol{U} \right) + \boldsymbol{U} \cdot \nabla \left(\frac{1}{2} \boldsymbol{U} \cdot \boldsymbol{U} \right)\\
&\overset{E = \frac{1}{2}\boldsymbol{U} \cdot \boldsymbol{U}}{=} \frac{\partial E}{\partial t} + \boldsymbol{U} \cdot \nabla E = \frac{DE}{Dt} && (2)\\
&\overset{\nabla \cdot \boldsymbol{U} = 0}{=} \frac{\partial E}{\partial t} + \nabla \cdot \big(\boldsymbol{U} E\big) && (3)
\end{aligned}$$

With this result, we may derive the next relation following from the momentum equation

$$\begin{aligned}
\rho \frac{DU_j}{Dt} &= \frac{\partial \tau_{ij}}{\partial x_i} \\
\Leftrightarrow \rho U_j \frac{DU_j}{Dt} &= U_j \frac{\partial \tau_{ij}}{\partial x_i} \\
\overset{(1)}{\Leftrightarrow} \rho \frac{DE}{Dt} &= U_j \frac{\partial \tau_{ij}}{\partial x_i} \\
\Leftrightarrow \rho \frac{DE}{Dt} &= \frac{\partial}{\partial x_i}\big(U_j \tau_{ij}\big) - \tau_{ij} \frac{\partial U_j}{\partial x_i} \\
\Leftrightarrow \rho \frac{DE}{Dt} - \frac{\partial}{\partial x_i}\big(U_j \tau_{ij}\big) &= -\tau_{ij} \frac{\partial U_j}{\partial x_i} \qquad (4)
\end{aligned}$$

Since the stress tensor is symmetric

$$\tau_{ij} = \tau_{ji}$$

we can express the right-hand-side of (3) as

$$\tau_{ij} \frac{\partial U_j}{\partial x_i} = \frac{1}{2}\big(\tau_{ij} + \tau_{ji}\big) \frac{\partial U_j}{\partial x_i} = \frac{1}{2}\tau_{ij} \frac{\partial U_j}{\partial x_i} + \frac{1}{2}\tau_{ji} \frac{\partial U_j}{\partial x_i}$$

Switching the indices of the second summand yields

$$\begin{aligned}
\tau_{ij} \frac{\partial U_j}{\partial x_i} &= \frac{1}{2}\tau_{ij} \frac{\partial U_j}{\partial x_i} + \frac{1}{2}\tau_{ij} \frac{\partial U_i}{\partial x_j} = \tau_{ij} \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \\
&= \tau_{ij} S_{ij} \qquad (5)
\end{aligned}$$

Inserting this into (3) leads to

$$\rho \frac{DE}{Dt} - \frac{\partial}{\partial x_i}\big(U_j \tau_{ij}\big) = -S_{ij} \tau_{ij} \qquad (6)$$

The stress writes for a Newtonian fluid as

$$\tau_{ij} = -p\delta_{ij} + 2\rho\nu S_{ij} \tag{7}$$

Plugging this expression into (5) yields

$$\begin{aligned}
\rho\frac{DE}{Dt} - \frac{\partial}{\partial x_i}\Big(U_j\big(-p\delta_{ij} + 2\rho\nu S_{ij}\big)\Big) &= -S_{ij}\big(-p\delta_{ij} + 2\rho\nu S_{ij}\big) \\
\Leftrightarrow \rho\frac{DE}{Dt} - \frac{\partial}{\partial x_i}\big(-U_i p + 2\rho\nu U_j S_{ij}\big) &= S_{ii}p - 2\rho\nu S_{ij}S_{ij}
\end{aligned} \tag{8}$$

Recalling the definition of the rate of strain tensor S_{ij}:

$$\begin{aligned}
S_{ij} &= \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \\
\Rightarrow S_{ii} &= \frac{1}{2}\left(\frac{\partial U_i}{\partial x_i} + \frac{\partial U_i}{\partial x_i}\right) = \frac{1}{2}\big(\nabla \cdot \boldsymbol{U} + \nabla \cdot \boldsymbol{U}\big) \overset{\text{Incompr.}}{=} 0
\end{aligned} \tag{9}$$

This simplifies (7) to

$$\begin{aligned}
\rho\frac{DE}{Dt} - \frac{\partial}{\partial x_i}\big(-U_i p + 2\rho\nu U_j S_{ij}\big) &= -2\rho\nu S_{ij}S_{ij} \\
\Leftrightarrow \frac{DE}{Dt} - \frac{\partial}{\partial x_i}\left(-\frac{U_i p}{\rho} + 2\nu U_j S_{ij}\right) &= -2\nu S_{ij}S_{ij}
\end{aligned}$$

By introducing

$$T_i \coloneqq \frac{U_i p}{\rho} - 2\nu U_j S_{ij} \tag{10}$$

we immediately obtain the final result

$$\frac{DE}{Dt} + \frac{\partial T_i}{\partial x_i} = -2\nu S_{ij}S_{ij} \,.$$