Turbulent Flows
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Cambridge University Press (2000)

Solution to Exercise 5.20

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We can rewrite Eq.(2.35) in the alternative form

$$\frac{\mathrm{D}U_j}{\mathrm{D}t} = \frac{\partial U_j}{\partial t} + \frac{\partial (U_i U_j)}{\partial x_i} = \nu \nabla^2 U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j}. \tag{1}$$

Similarly, by using Eq. (4.7) and Eq.(4.9) we can rewrite Eq.(4.12) as

$$\langle \frac{\mathrm{D}U_j}{\mathrm{D}t} \rangle = \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i U_j \rangle}{\partial x_i} = \nu \nabla^2 \langle U_j \rangle - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_j}. \tag{2}$$

Subtracting Eq.(2) from Eq.(1) yields Eq. (5.137)

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}(U_i U_j - \langle U_i U_j \rangle) = \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}, \tag{3}$$

where $u_j = U_j - \langle U_j \rangle$ and $p' = p - \langle p \rangle$. Expanding $\frac{\partial}{\partial x_i}(U_i U_j - \langle U_i U_j \rangle)$ by using the continuity equation for incompressible flow we obtain

$$\begin{aligned} \frac{\partial}{\partial x_i}(U_i U_j - \langle U_i U_j \rangle) &= \frac{\partial}{\partial x_i}(u_i \langle U_j \rangle + u_j \langle U_i \rangle + u_i u_j - \langle u_i u_j \rangle) \\ &= u_i \frac{\partial \langle U_j \rangle}{\partial x_i} + \langle U_i \rangle \frac{\partial u_j}{\partial x_i} + u_i \frac{\partial u_j}{\partial x_i} - \frac{\partial \langle u_i u_j \rangle}{\partial x_i} \end{aligned} \tag{4}$$

Substituting Eq.(4) into Eq.(3) and reorganizing the new equation we obtain Eq. (5.138)

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = -u_i \frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\partial}{\partial x_i}\langle u_i u_j \rangle + \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}. \tag{5}$$

where

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = \frac{\partial u_j}{\partial t} + (\langle U_i \rangle + u_i)\frac{\partial u_j}{\partial x_i}. \tag{6}$$

Substituting Eq.(6) into Eq.(5) and multiplying by u_j we obtain

$$u_j \frac{\partial u_j}{\partial t} + (\langle U_i \rangle + u_i) u_j \frac{\partial u_j}{\partial x_i}$$
$$= -u_i u_j \frac{\partial \langle U_j \rangle}{\partial x_i} + u_j \frac{\partial}{\partial x_i} \langle u_i u_j \rangle + \nu u_j \nabla^2 u_j - u_j \frac{1}{\rho} \frac{\partial p'}{\partial x_j}. \tag{7}$$

Simplifying this equation by using the continuity equation $\frac{\partial u_i}{\partial x_i} = 0$ we obtain

$$\frac{\partial (\frac{1}{2} u_j u_j)}{\partial t} + \langle U_i \rangle \frac{\partial (\frac{1}{2} u_j u_j)}{\partial x_i} + \frac{\partial (\frac{1}{2} u_i u_j u_j)}{\partial x_i}$$
$$= -u_i u_j \frac{\partial \langle U_j \rangle}{\partial x_i} + u_j \frac{\partial}{\partial x_i} \langle u_i u_j \rangle + \nu u_j \nabla^2 u_j - \frac{1}{\rho} \frac{\partial (p' u_j)}{\partial x_j}. \tag{8}$$

Taking the mean of the above equation and using $\langle u_i \rangle = 0$ we obtain

$$\frac{\partial k}{\partial t} + \langle U_i \rangle \frac{\partial k}{\partial x_i} + \frac{\partial \langle \frac{1}{2} u_i u_j u_j \rangle}{\partial x_i} = -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} + \nu \langle u_j \nabla^2 u_j \rangle - \frac{1}{\rho} \frac{\partial \langle p' u_j \rangle}{\partial x_j}, \tag{9}$$

where $k = \frac{1}{2} \langle u_i u_i \rangle$.
From Exercise 5.26, Eq. (5.163), we have

$$\nu \langle u_j \nabla^2 u_j \rangle = 2\nu \frac{\partial}{\partial x_i} \langle u_j s_{ij} \rangle - \varepsilon. \tag{10}$$

Substituting Eq.(10) into Eq.(9) and rearranging the terms yields

$$\frac{\partial k}{\partial t} + \langle U_i \rangle \frac{\partial k}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\langle \frac{1}{2} u_i u_j u_j \rangle + \frac{\langle u_i p' \rangle}{\rho} - 2\nu \langle u_j s_{ij} \rangle \right)$$
$$= -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} - \varepsilon. \tag{11}$$

This can be written in the alternative form

$$\frac{\bar{\mathrm{D}} k}{\bar{\mathrm{D}} t} + \nabla \cdot \mathbf{T}' = \mathcal{P} - \varepsilon, \tag{12}$$

where

$$T_i' \equiv \frac{1}{2} \langle u_i u_j u_j \rangle + \langle u_i p' \rangle / \rho - 2\nu \langle u_j s_{ij} \rangle. \tag{13}$$