Turbulent Flows
Stephen B. Pope
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Solution to Exercise 5.25

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Starting with the definition of ε (Eq. (5.128)) we have,

$$\varepsilon \equiv 2\nu\langle s_{ij}s_{ij}\rangle. \tag{1}$$

Using the defintion of s_{ij} from Eq. (5.130) we have,

$$\begin{aligned}
s_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
s_{ij}s_{ij} &= \frac{1}{4}\left\{2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i}\right)\right\} \\
s_{ij}s_{ij} &= \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\}. \qquad (2)
\end{aligned}$$

Using Eq. (2) in Eq. (1) and using definition of $\tilde{\varepsilon}$ (Eq. (5.159)) and product rule of differentiation we obtain,

$$\begin{aligned}
\varepsilon &= \tilde{\varepsilon} + \nu\left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle \\
&= \tilde{\varepsilon} + \nu\left\langle\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_j}{\partial x_i}\right) - u_i\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right\rangle.
\end{aligned}$$

Using the fact that $\nabla\cdot\mathbf{u} = 0$ for a constant property flow and the product rule of differentiation in the above equation,

$$\varepsilon = \tilde{\varepsilon} + \nu\left\langle\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_j}{\partial x_i}\right)\right\rangle$$

$$
\begin{aligned}
&= \tilde{\varepsilon} + \nu \left\langle \frac{\partial}{\partial x_j} \left(\frac{\partial (u_i u_j)}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_i} \right) \right\rangle \\
&= \tilde{\varepsilon} + \nu \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_i \partial x_j}. \qquad (3)
\end{aligned}
$$