Turbulent Flows
Stephen B. Pope
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Solution to Exercise 5.28

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In homogeneous isotropic turbulence the fourth-order tensor

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right\rangle \tag{1}$$

is isotropic, so that it can be written by using the Kronecker delta:

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right\rangle = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}. \tag{2}$$

Since the continuity equation gives $\partial u_i / \partial x_i = 0$, we have

$$\left\langle \frac{\partial u_i}{\partial x_i} \frac{\partial u_k}{\partial x_l} \right\rangle = 0, \tag{3}$$

which can be rewritten, using (1), as

$$\alpha \delta_{ii} \delta_{kl} + \beta \delta_{ik} \delta_{il} + \gamma \delta_{il} \delta_{ik} = 0, \tag{4}$$

or

$$(3\alpha + \gamma + \beta) \delta_{kl} = 0. \tag{5}$$

This leads to the following condition on α, β, γ:

$$(3\alpha + \gamma + \beta) = 0. \tag{6}$$

Because we are considering homogeneous turbulence, we have

$$\frac{\partial}{\partial x_j} \left\langle u_i \frac{\partial u_j}{\partial x_l} \right\rangle = 0 \tag{7}$$

and since

$$\frac{\partial}{\partial x_j}\left\langle u_i \frac{\partial u_j}{\partial x_l}\right\rangle = \left\langle u_i \frac{\partial^2 u_j}{\partial x_l \partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_l}\right\rangle, \tag{8}$$

with the first term in the RHS being zero on account of continuity, we find

$$\frac{\partial}{\partial x_j}\left\langle u_i \frac{\partial u_j}{\partial x_l}\right\rangle = 0. \tag{9}$$

Using (1), we can rewrite this equation as

$$\alpha\, \delta_{ij}\delta_{jl} + \beta\, \delta_{ij}\delta_{jl} + \gamma\, \delta_{il}\delta_{jj} = 0, \tag{10}$$

which also gives a condition on α, β, and γ:

$$(\alpha + \beta + 3\gamma) = 0. \tag{11}$$

Substituting (11) from (6) we find that $\alpha = \gamma$ and $\beta = -4\gamma$, or

$$\alpha = -\frac{\beta}{4}. \tag{12}$$

Replacing α and γ in (1) we obtain

$$\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right\rangle = \beta\left(\delta_{ik}\delta_{jl} - \frac{1}{4}\delta_{ij}\delta_{kl} - \frac{1}{4}\delta_{il}\delta_{jk}\right). \tag{13}$$

Finally using (13) with the appropriate indices, we find the following results:

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle = \frac{\beta}{2}, \quad \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle = \beta, \tag{14}$$

$$\left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right\rangle = \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle = -\frac{\beta}{4}, \tag{15}$$

or

$$\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle \quad = \quad 2\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle, \tag{16}$$

$$\left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right\rangle = \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle \quad = \quad -\frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle. \tag{17}$$

Since $\epsilon = \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle$, we also have

$$\epsilon = \nu\beta \left(\delta_{ii}\delta_{jj} - \frac{1}{4}\delta_{ij}\delta_{ij} - -\frac{1}{4}\delta_{ij}\delta_{ij} \right) = \nu\beta \left(9 - \frac{3}{4} - \frac{3}{4} \right), \tag{18}$$

or

$$\epsilon = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle . \tag{19}$$