Turbulent Flows

Stephen B. Pope Cambridge University Press (2000)

Solution to Exercise 5.28

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In homogeneous isotropic turbulence the fourth-order tensor

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right\rangle \tag{1}$$

is isotropic, so that it can be written by using the Kronecker delta:

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right\rangle = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}.$$
 (2)

Since the continuity equation gives $\partial u_i / \partial x_i = 0$, we have

$$\left\langle \frac{\partial u_i}{\partial x_i} \frac{\partial u_k}{\partial x_l} \right\rangle = 0, \tag{3}$$

which can be rewritten, using (1), as

$$\alpha \delta_{ii} \delta_{kl} + \beta \delta_{ik} \delta_{il} + \gamma \delta_{il} \delta_{ik} = 0, \tag{4}$$

or

$$(3\alpha + \gamma + \beta)\delta_{kl} = 0. \tag{5}$$

This leads to the following condition on α , β , γ :

$$(3\alpha + \gamma + \beta) = 0. \tag{6}$$

Because we are considering homogeneous turbulence, we have

$$\frac{\partial}{\partial x_j} \left\langle u_i \frac{\partial u_j}{\partial x_l} \right\rangle = 0 \tag{7}$$

and since

$$\frac{\partial}{\partial x_j} \left\langle u_i \frac{\partial u_j}{\partial x_l} \right\rangle = \left\langle u_i \frac{\partial^2 u_j}{\partial x_l \partial x_j} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_l} \right\rangle,\tag{8}$$

with the first term in the RHS being zero on account of continuity, we find

$$\frac{\partial}{\partial x_j} \left\langle u_i \frac{\partial u_j}{\partial x_l} \right\rangle = 0. \tag{9}$$

Using (1), we can rewrite this equation as

$$\alpha \,\delta_{ij}\delta_{jl} + \beta \,\delta_{ij}\delta_{jl} + \gamma \,\delta_{il}\delta_{jj} = 0, \tag{10}$$

which also gives a condition on α , β , and γ :

$$(\alpha + \beta + 3\gamma) = 0. \tag{11}$$

Substituting (11) from (6) we find that $\alpha = \gamma$ and $\beta = -4\gamma$, or

$$\alpha = -\frac{\beta}{4}.\tag{12}$$

Replacing α and γ in (1) we obtain

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right\rangle = \beta \left(\delta_{ik} \delta_{jl} - \frac{1}{4} \delta_{ij} \delta_{kl} - \frac{1}{4} \delta_{il} \delta_{jk} \right).$$
(13)

Finally using (13) with the appropriate indices, we find the following results:

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \frac{\beta}{2}, \quad \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = \beta,$$
 (14)

$$\left\langle \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \right\rangle = \left\langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right\rangle = -\frac{\beta}{4},\tag{15}$$

or

$$\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = 2 \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle,$$
 (16)

$$\left\langle \left(\frac{\partial u_1}{\partial x_2}\right) \right\rangle = 2\left\langle \left(\frac{\partial u_1}{\partial x_1}\right) \right\rangle, \tag{16}$$
$$\left\langle \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \right\rangle = \left\langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right\rangle = -\frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{17}$$

Since $\epsilon = \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle$, we also have

$$\epsilon = \nu\beta \left(\delta_{ii}\delta_{jj} - \frac{1}{4}\delta_{ij}\delta_{ij} - -\frac{1}{4}\delta_{ij}\delta_{ij}\right) = \nu\beta \left(9 - \frac{3}{4} - \frac{3}{4}\right),\tag{18}$$

$$\epsilon = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{19}$$

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