Turbulent Flows
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Solution to Exercise 5.41

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a) The Reynolds equations (Eq.(4.12)) are

$$\frac{\bar{\mathrm{D}}\langle U_j\rangle}{\bar{\mathrm{D}}t} = \nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}. \tag{1}$$

By differentiating Eq. 1 with respect to x_k, in homogeneous turbulence, we obtain

$$\frac{\partial}{\partial x_k}\left(\frac{\bar{\mathrm{D}}\langle U_j\rangle}{\bar{\mathrm{D}}t}\right) = \frac{\partial}{\partial x_k}\left(\nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i}\right) - \frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}\right), \tag{2}$$

i.e.,

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(\frac{\partial\langle U_j\rangle}{\partial t} + \langle U_i\rangle\frac{\partial\langle U_j\rangle}{\partial x_i}\right) &= \frac{\partial}{\partial x_k}\left(\nu\nabla^2\langle U_j\rangle\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial\langle u_i u_j\rangle}{\partial x_i}\right) \\ &\quad -\frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}\right).\end{aligned} \tag{3}$$

In Eq. 3, $\frac{\partial}{\partial x_k}\left(\nu\nabla^2\langle U_j\rangle\right)$ is zero because the mean velocity gradient is uniform for homogeneous turbulence and $\frac{\partial}{\partial x_k}\left(\frac{\partial\langle u_i u_j\rangle}{\partial x_i}\right)$ is zero because of homogeneity. So Eq. 3 reduces to

$$\frac{\partial}{\partial t}\frac{\partial\langle U_j\rangle}{\partial x_k} + \frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} + \langle U_i\rangle\frac{\partial^2\langle U_j\rangle}{\partial x_k\partial x_k} = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_k\partial x_j}. \tag{4}$$

Following the same argument, $\langle U_i\rangle\frac{\partial^2\langle U_j\rangle}{\partial x_k\partial x_k}$ is zero, so Eq. 4 reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\langle U_j\rangle}{\partial x_k} + \frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k}. \tag{5}$$

b) According to the definition of the mean rate of strain, we get

$$\bar{S}_{jk} \equiv \frac{1}{2}\left(\frac{\partial\langle U_j\rangle}{\partial x_k} + \frac{\partial\langle U_k\rangle}{\partial x_j}\right). \tag{6}$$

According to Eq. 5, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\langle U_j\rangle}{\partial x_k} + \frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k}, \tag{7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\langle U_k\rangle}{\partial x_j} + \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_k\rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k}. \tag{8}$$

Adding Eq. 7 to Eq. 8, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\langle U_j\rangle}{\partial x_k} + \frac{\partial\langle U_k\rangle}{\partial x_j}\right) + \frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_k\rangle}{\partial x_i} = -\frac{2}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k}, \tag{9}$$

i.e.,

$$\frac{\mathrm{d}\bar{S}_{jk}}{\mathrm{d}t} + \frac{1}{2}\frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{1}{2}\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_k\rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k}. \tag{10}$$

From Eq. 10, we get

$$\begin{aligned}\frac{\mathrm{d}\bar{S}_{jk}}{\mathrm{d}t} &+ \frac{1}{4}\left(\frac{\partial\langle U_i\rangle}{\partial x_k} + \frac{\partial\langle U_k\rangle}{\partial x_i}\right)\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right) \\ &+ \frac{1}{4}\left(\frac{\partial\langle U_i\rangle}{\partial x_k} - \frac{\partial\langle U_k\rangle}{\partial x_i}\right)\left(\frac{\partial\langle U_j\rangle}{\partial x_i} - \frac{\partial\langle U_i\rangle}{\partial x_j}\right) = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k},\end{aligned} \tag{11}$$

i.e.,

$$\frac{\mathrm{d}\bar{S}_{jk}}{\mathrm{d}t} + \bar{S}_{ik}\bar{S}_{ji} + \bar{\Omega}_{ik}\bar{\Omega}_{ji} = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x_j\partial x_k}, \tag{12}$$

c) According to the definition of the mean rate of rotation, we get

$$\bar{\Omega}_{jk} \equiv \frac{1}{2}\left(\frac{\partial\langle U_j\rangle}{\partial x_k} - \frac{\partial\langle U_k\rangle}{\partial x_j}\right). \tag{13}$$

Subtracting Eq. 8 from Eq. 7, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\langle U_j\rangle}{\partial x_k} - \frac{\partial\langle U_k\rangle}{\partial x_j}\right) + \frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} - \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_k\rangle}{\partial x_i} = 0, \tag{14}$$

i.e.,

$$\frac{\mathrm{d}\bar{\Omega}_{jk}}{\mathrm{d}t} + \frac{1}{2}\frac{\partial\langle U_i\rangle}{\partial x_k}\frac{\partial\langle U_j\rangle}{\partial x_i} - \frac{1}{2}\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_k\rangle}{\partial x_i} = 0. \tag{15}$$

From Eq. 15, we get

$$\begin{aligned}&\frac{\mathrm{d}\bar{\Omega}_{jk}}{\mathrm{d}t} + \frac{1}{4}\left(\frac{\partial\langle U_i\rangle}{\partial x_k} + \frac{\partial\langle U_k\rangle}{\partial x_i}\right)\left(\frac{\partial\langle U_j\rangle}{\partial x_i} - \frac{\partial\langle U_i\rangle}{\partial x_j}\right)\\ &+\frac{1}{4}\left(\frac{\partial\langle U_i\rangle}{\partial x_k} - \frac{\partial\langle U_k\rangle}{\partial x_i}\right)\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right) = 0,\end{aligned} \tag{16}$$

i.e.,

$$\frac{\mathrm{d}\bar{\Omega}_{jk}}{\mathrm{d}t} + \bar{S}_{ik}\bar{\Omega}_{ji} + \bar{\Omega}_{ik}\bar{S}_{ji} = 0. \tag{17}$$

[Note that the mean pressure field is of the form $\langle p(\mathbf{x},t)\rangle = A(t) + B_i(t)x_i + C_{ij}(t)x_i x_j$, and that $C_{ij}(t)$ can be chosen to produce any desired evolution $\mathrm{d}\bar{S}_{jk}/\mathrm{d}t$. On the other hand, the evolution of the mean rotation rate is entirely determined by $\bar{S}_{ij}$ and $\bar{\Omega}_{ij}$.]