Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.10

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We begin by using the *Kárman-Howarth* equation for the final period as,

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(u'^2 f\right) = \frac{2\nu u'^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \\
\Rightarrow \quad & f\frac{\mathrm{d}u'^2}{\mathrm{d}t} + u'^2\frac{\partial f}{\partial t} = \frac{2\nu u'^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \\
\Rightarrow \quad & \left(\frac{\mathrm{d}u'^2}{\mathrm{d}t}\frac{1}{u'^2}\right) f + \frac{\partial f}{\partial t} = \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \\
\Rightarrow \quad & \frac{\mathrm{d}u'^2}{\mathrm{d}t}\frac{1}{u'^2} = \frac{\frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) - \frac{\partial f}{\partial t}}{f} \qquad (1)
\end{aligned}$$

It is obvious that **LHS** of Eq.1 is only a function of t. Thus Eq. (6.93) can only satisfy Kárman-Howarth equation in the case **RHS** is also only a function of t. Inserting Eq. (6.93) into the **RHS** of Eq.1 yields,

$$\begin{aligned}
\mathbf{RHS} &= \frac{\frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\left(\frac{-2r}{8\nu t}\right)f\right) - \frac{r^2}{8\nu t^2}f}{f} \\
&= -\frac{\frac{1}{r^4}\left(\frac{5r^4}{2t}f - \frac{r^5}{2t}\frac{2r}{8\nu t}f\right) + \frac{r^2}{8\nu t}f}{f} \\
&= \frac{-5}{2}\,t \qquad (2)
\end{aligned}$$

Thus Eq. (6.93) satisfies the Kárman-Howarth equation. We also note that,

$$\begin{aligned}
& \frac{\mathrm{d}u'^2}{\mathrm{d}t}\frac{1}{u'^2} = \frac{-5}{2}\,t \\
\Rightarrow \quad & \frac{\mathrm{d}u'^2}{u'^2} = \frac{-5}{2}\frac{\mathrm{d}t}{t}
\end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & u'^2(t) = C\, t^{-5/2} \\ \Rightarrow \quad & k(t) = C'\, t^{-5/2} \end{aligned} \tag{3}$$