

Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.10

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We begin by using the *Kármán-Howarth* equation for the final period as,

$$\begin{aligned}
& \frac{\partial}{\partial t} (u'^2 f) = \frac{2\nu u'^2}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial f}{\partial r} \right) \\
\Rightarrow & f \frac{du'^2}{dt} + u'^2 \frac{\partial f}{\partial t} = \frac{2\nu u'^2}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial f}{\partial r} \right) \\
\Rightarrow & \left(\frac{du'^2}{dt} \frac{1}{u'^2} \right) f + \frac{\partial f}{\partial t} = \frac{2\nu}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial f}{\partial r} \right) \\
\Rightarrow & \frac{du'^2}{dt} \frac{1}{u'^2} = \frac{\frac{2\nu}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial f}{\partial r} \right) - \frac{\partial f}{\partial t}}{f} \tag{1}
\end{aligned}$$

It is obvious that **LHS** of Eq.1 is only a function of t . Thus Eq. (6.93) can only satisfy Kármán-Howarth equation in the case **RHS** is also only a function of t . Inserting Eq. (6.93) into the **RHS** of Eq.1 yields,

$$\begin{aligned}
\mathbf{RHS} &= \frac{\frac{2\nu}{r^4} \frac{\partial}{\partial r} \left(r^4 \left(\frac{-2r}{8\nu t} \right) f \right) - \frac{r^2}{8\nu t^2} f}{f} \\
&= -\frac{\frac{1}{r^4} \left(\frac{5r^4}{2t} f - \frac{r^5}{2t} \frac{2r}{8\nu t} f \right) + \frac{r^2}{8\nu t} f}{f} \\
&= \frac{-5}{2} t \tag{2}
\end{aligned}$$

Thus Eq. (6.93) satisfies the Kármán-Howarth equation. We also note that,

$$\begin{aligned}
& \frac{du'^2}{dt} \frac{1}{u'^2} = \frac{-5}{2} t \\
\Rightarrow & \frac{du'^2}{u'^2} = \frac{-5}{2} \frac{dt}{t}
\end{aligned}$$

$$\begin{aligned} \Rightarrow \quad u^2(t) &= C t^{-5/2} \\ \Rightarrow \quad k(t) &= C' t^{-5/2} \end{aligned} \tag{3}$$

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