

Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 6.12

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Date: 2/28/06

Let us consider the situation where $n = 0$,

$$\int_0^{\mathcal{L}} e^{2\pi i n x / \mathcal{L}} dx = \int_0^{\mathcal{L}} e^0 dx = \mathcal{L}. \quad (1)$$

While $n \neq 0$ yields,

$$\begin{aligned} \int_0^{\mathcal{L}} e^{2\pi i n x / \mathcal{L}} dx &= \frac{\mathcal{L}}{2\pi i n} e^{2\pi i n x / \mathcal{L}} \Big|_0^{\mathcal{L}} \\ &= \frac{\mathcal{L}}{2\pi i n} (e^{i2\pi n} - 1) \\ &= \frac{\mathcal{L}}{2\pi i n} (\cos(2\pi n) + i \sin(2\pi n) - 1) \\ &= 0. \end{aligned} \quad (2)$$

Using Eqs. (1) and (2), Eq. (6.112) may be written as,

$$\begin{aligned} \langle e^{i\boldsymbol{\kappa} \cdot \mathbf{x}} e^{-i\boldsymbol{\kappa}' \cdot \mathbf{x}} \rangle_{\mathcal{L}} &= \frac{1}{\mathcal{L}^3} \int_0^{\mathcal{L}} \int_0^{\mathcal{L}} \int_0^{\mathcal{L}} e^{i\boldsymbol{\kappa} \cdot \mathbf{x}} e^{-i\boldsymbol{\kappa}' \cdot \mathbf{x}} dx_1 dx_2 dx_3 \\ &= \frac{1}{\mathcal{L}^3} \prod_{j=1}^3 \int_0^{\mathcal{L}} e^{i\kappa_0 N_j x_j} dx_j \\ &= \frac{1}{\mathcal{L}^3} \prod_{j=1}^3 \int_0^{\mathcal{L}} e^{2\pi i N_j x_j / \mathcal{L}} dx_j \end{aligned} \quad (3)$$

Where $N_j = n_j - n'_j$ is an integer. It is now obvious that,

$$\begin{aligned} \langle e^{i\boldsymbol{\kappa} \cdot \mathbf{x}} e^{-i\boldsymbol{\kappa}' \cdot \mathbf{x}} \rangle_{\mathcal{L}} &= \begin{cases} 1, & N_j = 0 & \forall j \in \{1, 2, 3\} \\ 0, & N_j \neq 0 & \text{for some } j \in \{1, 2, 3\} \end{cases} \\ &= \delta_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'} \end{aligned} \quad (4)$$

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