Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.12

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Let us consider the situation where $n = 0$,

$$\int_0^{\mathcal{L}} e^{2\pi i n x/\mathcal{L}}\, \mathrm{d}x = \int_0^{\mathcal{L}} e^{0}\, \mathrm{d}x = \mathcal{L}. \tag{1}$$

While $n \neq 0$ yields,

$$\begin{aligned}
\int_0^{\mathcal{L}} e^{2\pi i n x/\mathcal{L}}\, \mathrm{d}x &= \frac{\mathcal{L}}{2\pi i n}\, e^{2\pi i n x/\mathcal{L}}\Big|_0^{\mathcal{L}} \\
&= \frac{\mathcal{L}}{2\pi i n}\left(e^{i2\pi n} - 1\right) \\
&= \frac{\mathcal{L}}{2\pi i n}(\cos(2\pi n) + i\sin(2\pi n) - 1) \\
&= 0.
\end{aligned} \tag{2}$$

Using Eqs. (1) and (2), Eq. (6.112) may be written as,

$$\begin{aligned}
\left\langle e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \right\rangle_{\mathcal{L}} &= \frac{1}{\mathcal{L}^3}\int_0^{\mathcal{L}}\int_0^{\mathcal{L}}\int_0^{\mathcal{L}} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 \\
&= \frac{1}{\mathcal{L}^3}\prod_{j=1}^{3}\int_0^{\mathcal{L}} e^{i\kappa_0 N_j x_j}\, \mathrm{d}x_j \\
&= \frac{1}{\mathcal{L}^3}\prod_{j=1}^{3}\int_0^{\mathcal{L}} e^{2\pi i N_j x_j/\mathcal{L}}\, \mathrm{d}x_j
\end{aligned} \tag{3}$$

Where $N_j = n_j - n'_j$ is an integer. It is now obvious that,

$$\begin{aligned}
\left\langle e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \right\rangle_{\mathcal{L}} &= \begin{cases} 1, & N_j = 0 \qquad \forall\, j \in \{1,2,3\} \\ 0, & N_j \neq 0 \qquad \text{for some}\, j \in \{1,2,3\} \end{cases} \\
&= \delta_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}
\end{aligned} \tag{4}$$