Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.19

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Using the definition of $R_{ij}(\boldsymbol{r})$ from Eq. (3.160) we obtain,

$$\begin{aligned} R_{ij}(\boldsymbol{r}) &= \langle R_{ij}(\boldsymbol{r})\rangle_{\mathcal{L}} \\ &= \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle_{\mathcal{L}} . \end{aligned} \quad (1)$$

Now, using Eq. (6.166) and Eq. (6.167) in Eq. (1) we get,

$$\begin{aligned} R_{ij}(\boldsymbol{r}) &= \left\langle \sum_{\boldsymbol{\kappa}'} e^{i\boldsymbol{\kappa}'\cdot\boldsymbol{x}}\hat{u}_i(\boldsymbol{\kappa}') \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot(\boldsymbol{x}+\boldsymbol{r})}\hat{u}_j(\boldsymbol{\kappa}) \right\rangle_{\mathcal{L}} \\ &= \sum_{\boldsymbol{\kappa}'}\sum_{\boldsymbol{\kappa}} \left\langle e^{i\boldsymbol{\kappa}'\cdot\boldsymbol{x}} e^{i\boldsymbol{\kappa}\cdot(\boldsymbol{x}+\boldsymbol{r})}\right\rangle_{\mathcal{L}} \left\langle \hat{u}_i(\boldsymbol{\kappa}')\hat{u}_j(\boldsymbol{\kappa})\right\rangle \\ &= \sum_{\boldsymbol{\kappa}'}\sum_{\boldsymbol{\kappa}} \left\langle e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} e^{i(\boldsymbol{\kappa}'+\boldsymbol{\kappa})\cdot\boldsymbol{x}}\right\rangle_{\mathcal{L}} \left\langle \hat{u}_i(\boldsymbol{\kappa}')\hat{u}_j(\boldsymbol{\kappa})\right\rangle \\ &= \sum_{\boldsymbol{\kappa}'}\sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \left\langle e^{i(\boldsymbol{\kappa}'+\boldsymbol{\kappa})\cdot\boldsymbol{x}}\right\rangle_{\mathcal{L}} \left\langle \hat{u}_i(\boldsymbol{\kappa}')\hat{u}_j(\boldsymbol{\kappa})\right\rangle \\ &= \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \left\langle \hat{u}_i(-\boldsymbol{\kappa})\hat{u}_j(\boldsymbol{\kappa})\right\rangle . \end{aligned} \quad (2)$$

Using conjugate symmetry (Eq. (6.166)) in Eq. (2) we obtain,

$$R_{ij}(\boldsymbol{r}) = \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \left\langle \hat{u}_i^*(\boldsymbol{\kappa})\hat{u}_j(\boldsymbol{\kappa})\right\rangle . \quad (3)$$

Now from the definition of $\hat{R}_{ij}(\boldsymbol{\kappa})$ (Eq. (6.152)) and Eq. (3) we obtain the final result of,

$$R_{ij}(\boldsymbol{r}) = \sum_{\boldsymbol{\kappa}} \hat{R}_{ij}(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} . \quad (4)$$