

Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 6.19

Prepared by: Sathyanarayana Ayyalasomayajula

Date: 03/04/03

Using the definition of $R_{ij}(\mathbf{r})$ from Eq. (3.160) we obtain,

$$\begin{aligned} R_{ij}(\mathbf{r}) &= \langle R_{ij}(\mathbf{r}) \rangle_{\mathcal{L}} \\ &= \langle u_i(\mathbf{x})u_j(\mathbf{x} + \mathbf{r}) \rangle_{\mathcal{L}}. \end{aligned} \quad (1)$$

Now, using Eq. (6.166) and Eq. (6.167) in Eq. (1) we get,

$$\begin{aligned} R_{ij}(\mathbf{r}) &= \left\langle \sum_{\boldsymbol{\kappa}'} e^{i\boldsymbol{\kappa}' \cdot \mathbf{x}} \hat{u}_i(\boldsymbol{\kappa}') \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa} \cdot (\mathbf{x} + \mathbf{r})} \hat{u}_j(\boldsymbol{\kappa}) \right\rangle_{\mathcal{L}} \\ &= \sum_{\boldsymbol{\kappa}'} \sum_{\boldsymbol{\kappa}} \left\langle e^{i\boldsymbol{\kappa}' \cdot \mathbf{x}} e^{i\boldsymbol{\kappa} \cdot (\mathbf{x} + \mathbf{r})} \right\rangle_{\mathcal{L}} \left\langle \hat{u}_i(\boldsymbol{\kappa}') \hat{u}_j(\boldsymbol{\kappa}) \right\rangle \\ &= \sum_{\boldsymbol{\kappa}'} \sum_{\boldsymbol{\kappa}} \left\langle e^{i\boldsymbol{\kappa} \cdot \mathbf{r}} e^{i(\boldsymbol{\kappa}' + \boldsymbol{\kappa}) \cdot \mathbf{x}} \right\rangle_{\mathcal{L}} \left\langle \hat{u}_i(\boldsymbol{\kappa}') \hat{u}_j(\boldsymbol{\kappa}) \right\rangle \\ &= \sum_{\boldsymbol{\kappa}'} \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa} \cdot \mathbf{r}} \left\langle e^{i(\boldsymbol{\kappa}' + \boldsymbol{\kappa}) \cdot \mathbf{x}} \right\rangle_{\mathcal{L}} \left\langle \hat{u}_i(\boldsymbol{\kappa}') \hat{u}_j(\boldsymbol{\kappa}) \right\rangle \\ &= \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa} \cdot \mathbf{r}} \left\langle \hat{u}_i(-\boldsymbol{\kappa}) \hat{u}_j(\boldsymbol{\kappa}) \right\rangle. \end{aligned} \quad (2)$$

Using conjugate symmetry (Eq. (6.166)) in Eq. (2) we obtain,

$$R_{ij}(\mathbf{r}) = \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa} \cdot \mathbf{r}} \left\langle \hat{u}_i^*(\boldsymbol{\kappa}) \hat{u}_j(\boldsymbol{\kappa}) \right\rangle. \quad (3)$$

Now from the definition of $\hat{R}_{ij}(\boldsymbol{\kappa})$ (Eq. (6.152)) and Eq. (3) we obtain the final result of,

$$R_{ij}(\mathbf{r}) = \sum_{\boldsymbol{\kappa}} \hat{R}_{ij}(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa} \cdot \mathbf{r}}. \quad (4)$$

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/1.0> or send a letter to Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.