Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.2

Prepared by: Daniel Doehring *Date:* 5/11/19

Differentiating the structure function $D_{ij}(\boldsymbol{r})$ yields

$$
\begin{aligned}
\frac{\partial D_{ij}}{\partial r_i} &\overset{(6.25)}{=} \frac{\partial}{\partial r_i}\left(D_{NN}(r,t)\delta_{ij} + \Big(D_{LL}(r,t) - D_{NN}(r,t)\Big)\frac{r_i r_j}{r^2}\right)\\
&= \frac{\partial D_{NN}}{\partial r_i}\delta_{ij} + \frac{\partial}{\partial r_i}\left(\Big(D_{LL} - D_{NN}\Big)\frac{r_i r_j}{r_k r_k}\right)\\
&= \frac{\partial D_{NN}}{\partial r_j} + \left(\frac{\partial D_{LL}}{\partial r_i} - \frac{\partial D_{NN}}{\partial r_i}\right)\frac{r_i r_j}{r_k r_k} + \Big(D_{LL} - D_{NN}\Big)\left[\frac{r_j\frac{\partial r_i}{\partial r_i} + r_i\frac{\partial r_j}{\partial r_i}}{r_k r_k} + r_i r_j\frac{\partial}{\partial r_i}(r_k r_k)^{-1}\right]\\
&= \frac{\partial D_{NN}}{\partial r}\frac{\partial r}{\partial r_j} + \left(\frac{\partial D_{LL}}{\partial r}\frac{\partial r}{\partial r_i} - \frac{\partial D_{NN}}{\partial r}\frac{\partial r}{\partial r_i}\right)\frac{r_i r_j}{r_k r_k}\\
&\quad + \Big(D_{LL} - D_{NN}\Big)\left(\frac{3r_j + r_i\delta_{ji}}{r_k r_k} - \frac{r_i r_j}{(r_k r_k)^2}\frac{\partial}{\partial r_i}r_l r_l\right)\\
&= \frac{\partial D_{NN}}{\partial r}\frac{\partial\sqrt{r_k r_k}}{\partial r_j} + \left(\frac{\partial D_{LL}}{\partial r}\frac{\partial\sqrt{r_l r_l}}{\partial r_i} - \frac{\partial D_{NN}}{\partial r}\frac{\partial\sqrt{r_l r_l}}{\partial r_i}\right)\frac{r_i r_j}{r_k r_k}\\
&\quad + \Big(D_{LL} - D_{NN}\Big)\left(\frac{3r_j + r_i\delta_{ji}}{r_k r_k} - \frac{r_i r_j}{(r_k r_k)^2}2\delta_{il}r_l\right)\\
&= \frac{\partial D_{NN}}{\partial r}\frac{1}{2\sqrt{r_k r_k}}2\delta_{jl}r_l + \left(\frac{\partial D_{LL}}{\partial r}\frac{1}{2\sqrt{r_l r_l}}2\delta_{im}r_m - \frac{r_i r_j}{r_l r_l}\frac{\partial D_{NN}}{\partial r}\frac{1}{2\sqrt{r_l r_l}}2\delta_{im}r_m\right)\frac{r_i r_j}{r_k r_k}\\
&\quad + \Big(D_{LL} - D_{NN}\Big)\left(\frac{3r_j + r_j}{r_k r_k} - 2\frac{r_i r_j}{(r_k r_k)^2}r_i\right)\\
&= \frac{\partial D_{NN}}{\partial r}\frac{r_j}{r} + \left(\frac{\partial D_{NN}}{\partial r}\frac{r_i}{r} - \frac{\partial D_{LL}}{\partial r_i}\frac{r_i}{r}\right)\frac{r_j r_i}{r_k r_k} + \Big(D_{LL} - D_{NN}\Big)\left(\frac{4r_j}{r_k r_k} - 2\frac{r_j}{r_k r_k}\right)\\
&= \frac{\partial D_{NN}}{\partial r}\frac{r_j}{r} + \frac{r_j r^2}{r^3}\frac{\partial D_{LL}}{\partial r} - \frac{r_j}{r}\frac{\partial D_{NN}}{\partial r} + \Big(D_{LL} - D_{NN}\Big)2\frac{r_j}{r^2}\\
&= \frac{r_j}{r^2}\left(r\frac{\partial D_{LL}}{\partial r} + 2\Big(D_{LL} - D_{NN}\Big)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial D_{ij}}{\partial r_i} &= \frac{r_j}{r^2}\left(r\frac{\partial D_{LL}}{\partial r} + 2\Big(D_{LL} - D_{NN}\Big)\right) \\
\overset{(6.37)}{\Leftrightarrow} 0 &= \frac{r_j}{r^2}\left(r\frac{\partial D_{LL}}{\partial r} + 2\Big(D_{LL} - D_{NN}\Big)\right) \\
\overset{r_j \neq 0}{\Leftrightarrow} 0 &= r\frac{\partial D_{LL}}{\partial r} + 2\Big(D_{LL} - D_{NN}\Big) \\
\Leftrightarrow D_{NN} &= \frac{r}{2}\frac{\partial D_{LL}}{\partial r} + D_{LL} \qquad (1)
\end{aligned}
$$

The equation stated above is the short-handed notation of equation (6.28).