

Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.2

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Differentiating the structure function $D_{ij}(\mathbf{r})$ yields

$$\begin{aligned}
\frac{\partial D_{ij}}{\partial r_i} &\stackrel{(6.25)}{=} \frac{\partial}{\partial r_i} \left(D_{NN}(r, t) \delta_{ij} + \left(D_{LL}(r, t) - D_{NN}(r, t) \right) \frac{r_i r_j}{r^2} \right) \\
&= \frac{\partial D_{NN}}{\partial r_i} \delta_{ij} + \frac{\partial}{\partial r_i} \left(\left(D_{LL} - D_{NN} \right) \frac{r_i r_j}{r_k r_k} \right) \\
&= \frac{\partial D_{NN}}{\partial r_j} + \left(\frac{\partial D_{LL}}{\partial r_i} - \frac{\partial D_{NN}}{\partial r_i} \right) \frac{r_i r_j}{r_k r_k} + \left(D_{LL} - D_{NN} \right) \left[\frac{r_j \frac{\partial r_i}{\partial r_i} + r_i \frac{\partial r_j}{\partial r_i}}{r_k r_k} + r_i r_j \frac{\partial}{\partial r_i} (r_k r_k)^{-1} \right] \\
&= \frac{\partial D_{NN}}{\partial r} \frac{\partial r}{\partial r_j} + \left(\frac{\partial D_{LL}}{\partial r} \frac{\partial r}{\partial r_i} - \frac{\partial D_{NN}}{\partial r} \frac{\partial r}{\partial r_i} \right) \frac{r_i r_j}{r_k r_k} \\
&\quad + \left(D_{LL} - D_{NN} \right) \left(\frac{3r_j + r_i \delta_{ji}}{r_k r_k} - \frac{r_i r_j}{(r_k r_k)^2} \frac{\partial}{\partial r_i} r_l r_l \right) \\
&= \frac{\partial D_{NN}}{\partial r} \frac{\partial \sqrt{r_k r_k}}{\partial r_j} + \left(\frac{\partial D_{LL}}{\partial r} \frac{\partial \sqrt{r_l r_l}}{\partial r_i} - \frac{\partial D_{NN}}{\partial r} \frac{\partial \sqrt{r_l r_l}}{\partial r_i} \right) \frac{r_i r_j}{r_k r_k} \\
&\quad + \left(D_{LL} - D_{NN} \right) \left(\frac{3r_j + r_i \delta_{ji}}{r_k r_k} - \frac{r_i r_j}{(r_k r_k)^2} 2\delta_{il} r_l \right) \\
&= \frac{\partial D_{NN}}{\partial r} \frac{1}{2\sqrt{r_k r_k}} 2\delta_{jl} r_l + \left(\frac{\partial D_{LL}}{\partial r} \frac{1}{2\sqrt{r_l r_l}} 2\delta_{im} r_m - \frac{r_i r_j}{r_l r_l} \frac{\partial D_{NN}}{\partial r} \frac{1}{2\sqrt{r_l r_l}} 2\delta_{im} r_m \right) \frac{r_i r_j}{r_k r_k} \\
&\quad + \left(D_{LL} - D_{NN} \right) \left(\frac{3r_j + r_j}{r_k r_k} - 2 \frac{r_i r_j}{(r_k r_k)^2} r_i \right) \\
&= \frac{\partial D_{NN}}{\partial r} \frac{r_j}{r} + \left(\frac{\partial D_{NN}}{\partial r} \frac{r_i}{r} - \frac{\partial D_{LL}}{\partial r_i} \frac{r_i}{r} \right) \frac{r_j r_i}{r_k r_k} + \left(D_{LL} - D_{NN} \right) \left(\frac{4r_j}{r_k r_k} - 2 \frac{r_j}{r_k r_k} \right) \\
&= \frac{\partial D_{NN}}{\partial r} \frac{r_j}{r} + \frac{r_j r^2}{r^3} \frac{\partial D_{LL}}{\partial r} - \frac{r_j}{r} \frac{\partial D_{NN}}{\partial r} + \left(D_{LL} - D_{NN} \right) 2 \frac{r_j}{r^2} \\
&= \frac{r_j}{r^2} \left(r \frac{\partial D_{LL}}{\partial r} + 2 \left(D_{LL} - D_{NN} \right) \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial D_{ij}}{\partial r_i} &= \frac{r_j}{r^2} \left(r \frac{\partial D_{LL}}{\partial r} + 2(D_{LL} - D_{NN}) \right) \\
\stackrel{(6.37)}{\Leftrightarrow} 0 &= \frac{r_j}{r^2} \left(r \frac{\partial D_{LL}}{\partial r} + 2(D_{LL} - D_{NN}) \right) \\
\stackrel{r_j \neq 0}{\Leftrightarrow} 0 &= r \frac{\partial D_{LL}}{\partial r} + 2(D_{LL} - D_{NN}) \\
\Leftrightarrow D_{NN} &= \frac{r}{2} \frac{\partial D_{LL}}{\partial r} + D_{LL} \tag{1}
\end{aligned}$$

The equation stated above is the short-handed notation of equation (6.28).

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