**Turbulent Flows**
Stephen B. Pope
*Cambridge University Press* (2000)

# Solution to Exercise 6.20

*Prepared by:* Sathyanarayana Ayyalasomayajula *Date:* 03/05/03

Using the definition of $\hat{R}_{ij}(\boldsymbol{\kappa})$ from Eq. (6.152) with $i = j$,

$$\hat{R}_{(i)(i)}(\boldsymbol{\kappa}) = \langle \hat{u}^*_{(i)}(\boldsymbol{\kappa}, t)\hat{u}_{(i)}(\boldsymbol{\kappa}, t)\rangle \geq 0,$$

whose bracketed suffixes are excluded from the summation convention. Then, applying the summination convention, we clearly obtain $\hat{R}_{ii}(\boldsymbol{\kappa}) \geq 0$.

Starting with $\hat{R}_{ji}(-\boldsymbol{\kappa})$ we have,

$$\begin{aligned}\hat{R}_{ji}(-\boldsymbol{\kappa}) &= \left\langle \hat{u}^*_j(-\boldsymbol{\kappa}, t)\hat{u}_i(-\boldsymbol{\kappa}, t)\right\rangle \\ &= \left\langle \hat{u}_j(\boldsymbol{\kappa}, t)\hat{u}^*_i(\boldsymbol{\kappa}, t)\right\rangle \\ &= \hat{R}_{ij}(\boldsymbol{\kappa}). \end{aligned} \quad (1)$$

Alternatively starting with $\hat{R}^*_{ji}(\boldsymbol{\kappa})$ we obtain,

$$\begin{aligned}\hat{R}^*_{ji}(\boldsymbol{\kappa}) &= \left\langle \hat{u}^*_j(\boldsymbol{\kappa}, t)\hat{u}_i(\boldsymbol{\kappa}, t)\right\rangle^* \\ &= \left\langle \hat{u}_j(\boldsymbol{\kappa}, t)\hat{u}^*_i(\boldsymbol{\kappa}, t)\right\rangle \\ &= \hat{R}_{ij}(\boldsymbol{\kappa}). \end{aligned} \quad (2)$$

Hence we have the result,

$$\hat{R}_{ij}(\boldsymbol{\kappa}) = \hat{R}_{ji}(-\boldsymbol{\kappa}) = \hat{R}^*_{ji}(\boldsymbol{\kappa}). \quad (3)$$

The orthogonality of $\boldsymbol{\kappa}$ of $\hat{\boldsymbol{R}}(\boldsymbol{\kappa})$ is shown as follows,

$$\begin{aligned}\kappa_i\hat{R}_{ij}(\boldsymbol{\kappa}) &= \kappa_i\left\langle \hat{u}^*_i(\boldsymbol{\kappa}, t)\hat{u}_j(\boldsymbol{\kappa}, t)\right\rangle \\ &= \left\langle \kappa_i\hat{u}^*_i(\boldsymbol{\kappa}, t)\hat{u}_j(\boldsymbol{\kappa}, t)\right\rangle \\ &= \left\langle \kappa_i\hat{u}_i(-\boldsymbol{\kappa}, t)\hat{u}_j(\boldsymbol{\kappa}, t)\right\rangle \\ &= -\left\langle (-\kappa_i)\hat{u}_i(-\boldsymbol{\kappa}, t)\hat{u}_j(\boldsymbol{\kappa}, t)\right\rangle \end{aligned} \quad (4)$$

Now replacing $-\boldsymbol{\kappa}$ with $\boldsymbol{\kappa}'$ and using conjugate symmetry and $\boldsymbol{\kappa}\cdot\hat{\boldsymbol{u}}(\boldsymbol{\kappa}) = 0$ (continuity) in Eq. (4) we obtain,

$$\kappa_i \hat{R}_{ij}(\boldsymbol{\kappa}) = -\left\langle \{\kappa'_i \hat{u}_i(\boldsymbol{\kappa}', t)\}\, \hat{u}^*_j(\boldsymbol{\kappa}', t) \right\rangle = 0. \tag{5}$$

Similarly,

$$\begin{aligned} \kappa_j \hat{R}_{ij}(\boldsymbol{\kappa}) &= \kappa_j \left\langle \hat{u}^*_i(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \right\rangle \\ &= \left\langle \kappa_j \hat{u}^*_i(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \right\rangle \\ &= \left\langle \hat{u}^*_i(\boldsymbol{\kappa}, t) \{\kappa_j \hat{u}_j(\boldsymbol{\kappa}, t)\} \right\rangle \\ &= 0. \end{aligned} \tag{6}$$