

Turbulent Flows
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Solution to Exercise 6.20

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Using the definition of $\hat{R}_{ij}(\boldsymbol{\kappa})$ from Eq. (6.152) with $i = j$,

$$\hat{R}_{(i)(i)}(\boldsymbol{\kappa}) = \langle \hat{u}_{(i)}^*(\boldsymbol{\kappa}, t) \hat{u}_{(i)}(\boldsymbol{\kappa}, t) \rangle \geq 0,$$

whose bracketed suffixes are excluded from the summation convention. Then, applying the summation convention, we clearly obtain $\hat{R}_{ii}(\boldsymbol{\kappa}) \geq 0$.

Starting with $\hat{R}_{ji}(-\boldsymbol{\kappa})$ we have,

$$\begin{aligned} \hat{R}_{ji}(-\boldsymbol{\kappa}) &= \langle \hat{u}_j^*(-\boldsymbol{\kappa}, t) \hat{u}_i(-\boldsymbol{\kappa}, t) \rangle \\ &= \langle \hat{u}_j(\boldsymbol{\kappa}, t) \hat{u}_i^*(\boldsymbol{\kappa}, t) \rangle \\ &= \hat{R}_{ij}(\boldsymbol{\kappa}). \end{aligned} \tag{1}$$

Alternatively starting with $\hat{R}_{ji}^*(\boldsymbol{\kappa})$ we obtain,

$$\begin{aligned} \hat{R}_{ji}^*(\boldsymbol{\kappa}) &= \langle \hat{u}_j^*(\boldsymbol{\kappa}, t) \hat{u}_i(\boldsymbol{\kappa}, t) \rangle^* \\ &= \langle \hat{u}_j(\boldsymbol{\kappa}, t) \hat{u}_i^*(\boldsymbol{\kappa}, t) \rangle \\ &= \hat{R}_{ij}(\boldsymbol{\kappa}). \end{aligned} \tag{2}$$

Hence we have the result,

$$\hat{R}_{ij}(\boldsymbol{\kappa}) = \hat{R}_{ji}(-\boldsymbol{\kappa}) = \hat{R}_{ji}^*(\boldsymbol{\kappa}). \tag{3}$$

The orthogonality of $\boldsymbol{\kappa}$ of $\hat{\mathbf{R}}(\boldsymbol{\kappa})$ is shown as follows,

$$\begin{aligned} \kappa_i \hat{R}_{ij}(\boldsymbol{\kappa}) &= \kappa_i \langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \rangle \\ &= \langle \kappa_i \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \rangle \\ &= \langle \kappa_i \hat{u}_i(-\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \rangle \\ &= -\langle (-\kappa_i) \hat{u}_i(-\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \rangle \end{aligned} \tag{4}$$

Now replacing $-\boldsymbol{\kappa}$ with $\boldsymbol{\kappa}'$ and using conjugate symmetry and $\boldsymbol{\kappa} \cdot \hat{\boldsymbol{u}}(\boldsymbol{\kappa}) = 0$ (continuity) in Eq. (4) we obtain,

$$\kappa_i \hat{R}_{ij}(\boldsymbol{\kappa}) = -\langle \{ \kappa'_i \hat{u}_i(\boldsymbol{\kappa}', t) \} \hat{u}_j^*(\boldsymbol{\kappa}', t) \rangle = 0. \quad (5)$$

Similarly,

$$\begin{aligned} \kappa_j \hat{R}_{ij}(\boldsymbol{\kappa}) &= \kappa_j \langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \rangle \\ &= \langle \kappa_j \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \rangle \\ &= \langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \{ \kappa_j \hat{u}_j(\boldsymbol{\kappa}, t) \} \rangle \\ &= 0. \end{aligned} \quad (6)$$

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