Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 6.35

Prepared by: Timo de Ruijsscher *Date:* 11/23/2016

In simple turbulent shear flow, the only non-zero mean velocity gradient is $\partial \langle U_i \rangle / \partial x_j$. Combining this knowledge with $\mathcal{P} \equiv - \langle u_i u_j \rangle \partial \langle U_i \rangle / \partial x_j$ (Eq.5.133), the production rate of turbulent kinetic energy is given by

$$\mathcal{P} \equiv - \langle u_1 u_2 \rangle \partial \langle U_1 \rangle / \partial x_2. \tag{1}$$

Substituting the given expressions $\mathcal{S} = \partial \langle U_1 \rangle / \partial x_2$ and $\alpha \equiv - \langle u_1 u_2 \rangle / k$ in $\mathcal{S}k/\varepsilon$, and using Eq.(1) it is shown that

$$\begin{aligned} \mathcal{S}k/\varepsilon &= -\frac{\partial \langle U_1 \rangle}{\partial x_2} \frac{\langle u_1 u_2 \rangle}{\alpha} \frac{1}{\varepsilon} \\ &= \frac{1}{\alpha} \frac{\mathcal{P}}{\varepsilon} \approx 3, \end{aligned} \tag{2}$$

where the numeric result is obtained by substituting the given values $\mathcal{P}/\epsilon \approx 1$ and $\alpha \approx 0.3$.

By using the Kolmogorov timescale $\tau_\eta = (\nu/\varepsilon)^{1/2}$ (Eq.6.3), Eq.(2) and the definition of the turbulence Reynolds number $\mathrm{Re}_L \equiv k^{1/2} L/\nu$ (Eq.6.59) it is shown that

$$\begin{aligned} \mathcal{S}\tau_\eta &= \frac{1}{\alpha} \frac{\mathcal{P}}{\varepsilon} \frac{\varepsilon}{k} \left(\frac{\nu}{\varepsilon} \right)^{1/2} \\ &= \frac{1}{\alpha} \frac{\mathcal{P}}{\varepsilon} \left(\frac{\varepsilon \nu}{k^2} \right)^{-1/2} \\ &= \frac{1}{\alpha} \frac{\mathcal{P}}{\varepsilon} \left(\mathrm{Re}_L \right)^{-1/2} \approx 3 \mathrm{Re}_L^{-1/2}. \end{aligned} \tag{3}$$

When the relation between the turbulence and the Taylor-scale Reynolds numbers $\mathrm{R}_\lambda = \left(\frac{20}{3} \mathrm{Re}_L \right)^{1/2}$ (Eq.6.64) is substituted in the result from Eq.(3), it is obtained that

$$\begin{aligned}
\mathcal{S}\tau_\eta &= \frac{1}{\alpha}\frac{\mathcal{P}}{\varepsilon}\left(\mathrm{Re}_L\right)^{-1/2} \\
&= \frac{1}{\alpha}\frac{\mathcal{P}}{\varepsilon}\left(\frac{3}{20}\mathrm{R}_\lambda^2\right)^{-1/2} \\
&= \sqrt{\frac{20}{3}}\frac{1}{\alpha}\frac{\mathcal{P}}{\varepsilon}\mathrm{R}_\lambda^{-1} \approx 9\mathrm{R}_\lambda^{-1}.
\end{aligned} \tag{4}$$

Using the definition of the length scale $L_\mathcal{S}$ from Eq.(6.272), the solution of Eq.(2) and the definition of the lengthscale $L \equiv k^{3/2}/\varepsilon$ characterizing the large eddies, it is shown that

$$\begin{aligned}
L_\mathcal{S} &\equiv \mathcal{S}^{-3/2}\varepsilon^{1/2} \\
&= \left(\frac{\mathcal{P}}{\alpha k}\right)^{-3/2}\varepsilon^{1/2} \\
&= \frac{\mathcal{P}^{-3/2}}{\alpha^{-3/2}}\varepsilon^{1/2}L\varepsilon \\
&= \left(\frac{\mathcal{P}}{\varepsilon}\right)^{-3/2}\alpha^{3/2}L \approx \frac{1}{6}L.
\end{aligned} \tag{5}$$

Finally, by substituting the result of Eq.(2) in the definition of the velocity scale $u_\mathcal{S}$ from Eq.(6.276), it is shown that

$$\begin{aligned}
u_\mathcal{S} &\equiv (\varepsilon/\mathcal{S})^{1/2} \\
&= \left(\frac{\varepsilon\alpha k}{\mathcal{P}}\right)^{1/2} \\
&= \alpha^{1/2}\left(\frac{\mathcal{P}}{\varepsilon}\right)^{1/2}k^{1/2} \approx \frac{1}{2}k^{1/2}.
\end{aligned} \tag{6}$$