Turbulent Flows
Stephen B. Pope
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Solution to Exercise 6.36

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Substituting $\mathcal{T}_\kappa(\kappa) = E(\kappa)\alpha^{-1}\varepsilon^{1/3}\kappa^{5/3}$ (Eq.6.298) into Eq.(6.296) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\kappa}\mathcal{T}_\kappa(\kappa) &= -2\nu\kappa^2 E(\kappa) \\ \frac{\mathrm{d}}{\mathrm{d}\kappa}\left(E(\kappa)\alpha^{-1}\varepsilon^{1/3}\kappa^{5/3}\right) &= -2\nu\kappa^2 E(\kappa) \\ \frac{1}{E(\kappa)\kappa^{5/3}}\frac{\mathrm{d}}{\mathrm{d}\kappa}\left(E(\kappa)\kappa^{5/3}\right) &= -2\nu\alpha\varepsilon^{-1/3}\kappa^{1/3}. \qquad (1) \end{aligned}$$

Now using the differentiation rule for the natural logarithm, Eq.(1) can be solved for $E(\kappa)$ by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\kappa}\ln\left(E(\kappa)\kappa^{5/3}\right) &= -2\alpha\nu\varepsilon^{-1/3}\kappa^{1/3} \\ \ln\left(E(\kappa)\kappa^{5/3}\right) &= -\frac{3}{2}\alpha\nu\varepsilon^{-1/3}\kappa^{4/3} + \hat{\beta} \\ E(\kappa) &= \beta\kappa^{-5/3}\exp\left(-\frac{3}{2}\alpha\nu\varepsilon^{-1/3}\kappa^{4/3}\right) \\ &= \beta\kappa^{-5/3}\exp\left[-\frac{3}{2}\alpha\left(\kappa\eta\right)^{4/3}\right], \qquad (2) \end{aligned}$$

where $\hat{\beta} = \exp(\beta)$ is a constant of integration. According to the Kolmogorov spectrum, $E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3}$ for small $\kappa\eta$, which eventually leads to

$$\lim_{\kappa\eta\to 0} E(\kappa) = \beta\kappa^{-5/3} \equiv C\varepsilon^{2/3}\kappa^{-5/3}, \qquad (3)$$

from which it is concluded that $\boxed{\beta = C\varepsilon^{2/3}}$.

The result from Eq.(2) is used to calculate the dissipation as

$$
\begin{aligned}
\epsilon &= \int_0^\infty 2\nu\kappa^2 E(\kappa)\mathrm{d}\kappa \\
&= \int_0^\infty 2\nu\kappa^2\beta\kappa^{-5/3}\exp\left[-\frac{3}{2}\alpha\left(\kappa\eta\right)^{4/3}\right]\mathrm{d}\kappa \\
&= 2\nu\beta\int_0^\infty \kappa^{1/3}\exp\left[-\frac{3}{2}\alpha\left(\kappa\eta\right)^{4/3}\right]\mathrm{d}\kappa \\
&= 2\nu\beta\left[-\frac{3}{4}\frac{1}{\frac{3}{2}\alpha\eta^{4/3}}\exp\left[-\frac{3}{2}\alpha\left(\kappa\eta\right)^{4/3}\right]\right]_0^\infty \\
&= \frac{\nu\beta}{\alpha\eta^{4/3}} \\
&= \frac{\varepsilon^{1/3}\beta}{\alpha},
\end{aligned} \tag{4}
$$

where in the last step Eq.(6.1) is used, and the integral is calculated using the identity $\int x^{a-1}\exp\left[bx^a\right]\mathrm{d}x = \frac{1}{ab}\exp\left[bx^a\right]$ (with a and b arbitrary constants).

Using the result $\beta = C\varepsilon^{2/3}$ from above and substituting it in the result of Eq.(4), it is obtained that $\varepsilon = \varepsilon C/\alpha$ and thus that $\boxed{\alpha = C}$. Now substituting both $\alpha = C$ and $\beta = C\varepsilon^{2/3}$ in Eq.(6.301) yields

$$
\begin{aligned}
E(\kappa) &= \beta\kappa^{-5/3}\exp\left[-\frac{3}{2}\alpha\left(\kappa\eta\right)^{4/3}\right] \\
&= C\varepsilon^{2/3}\kappa^{-5/3}\exp\left[-\frac{3}{2}C\left(\kappa\eta\right)^{4/3}\right],
\end{aligned} \tag{5}
$$

which is the equation of the Pao spectrum (Eq.6.299).

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