Solution to Exercise 6.6

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a) From Eq.(6.46),

$$g(r,t) = f(r,t) + \frac{1}{2}r\frac{\partial}{\partial r}f(r,t), \tag{1}$$

we have

$$\begin{aligned} g'(r,t) &= f'(r,t) + \frac{1}{2}f'(r,t) + \frac{1}{2}rf''(r,t) \\ &= \frac{3}{2}f'(r,t) + \frac{1}{2}rf''(r,t), \end{aligned} \tag{2}$$

so

$$\begin{aligned} g''(r,t) &= \frac{3}{2}f''(r,t) + \frac{1}{2}f''(r,t) + \frac{1}{2}rf'''(r,t) \\ &= 2f''(r,t) + \frac{1}{2}rf'''(r,t). \end{aligned} \tag{3}$$

b) According to Eq.(6.57)

$$\begin{aligned} \lambda_g(t) &= \left[-\frac{1}{2}g''(0)\right]^{-\frac{1}{2}} \\ &= \left\{-\frac{1}{2}\left[2f''(0,t) + \frac{1}{2}rf'''(0,t)\right]\right\}^{-\frac{1}{2}} \\ &= \left[-f''(0)\right]^{-\frac{1}{2}} \\ &= \frac{\lambda_f(t)}{\sqrt{2}}. \end{aligned} \tag{4}$$

Notice that $f'''(0,t) = 0$ because $f(r)$ is an even function.

c) From Eqs.(6.56) and (6.68) we have

$$\left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle = \frac{2u'^2}{{\lambda_f}^2} = \frac{u'^2}{{\lambda_g}^2}, \tag{5}$$

and hence from Eq.(5.169)

$$\left\langle \left(\frac{\partial u_1}{\partial x_2} \right)^2 \right\rangle = 2 \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle, \tag{6}$$

we obtain

$$\left\langle \left(\frac{\partial u_1}{\partial x_2} \right)^2 \right\rangle = \frac{2u'^2}{{\lambda_g}^2}. \tag{7}$$