Turbulent Flows
Stephen B. Pope
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Solution to Exercise 7.11

Prepared by: Michael Stumpf *Date:* 12/17/17

In fully developed turbulent pipe flow, the mean temporal derivatives vanish and velocity statistics only depend on the radial coordinate. Hence, Eq. (5.45) reduces to

$$\frac{\partial}{\partial r}(r\langle V\rangle) = 0,$$

which, with the boundary condition $\langle V\rangle_{r=R}$, integrates to

$$\langle V\rangle = 0.$$

The radial momentum equation (5.47) then simplifies to

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\langle v^2\rangle) + \frac{\langle w^2\rangle}{r} \\ &= -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial r} - \frac{\partial\langle v^2\rangle}{\partial r} + \frac{\langle w^2\rangle - \langle v^2\rangle}{r}. \end{aligned}$$

Differentiating in the streamwise direction yields

$$0 = -\frac{1}{\rho}\frac{\partial^2\langle p\rangle}{\partial x\partial r},$$

and thus

$$\frac{\partial\langle p\rangle}{\partial x} = \frac{\mathrm{d}p_w}{\mathrm{d}x} = \text{const.}$$

The axial momentum equation (5.46) simplifies to

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x} - \frac{1}{r}\frac{\partial}{\partial r}(r\langle uv\rangle) + \nu\frac{\partial^2\langle U\rangle}{\partial r^2} \\ &= -r\frac{\mathrm{d}p_w}{\mathrm{d}x} - \rho\frac{\partial}{\partial r}(r\langle uv\rangle) + r\nu\rho\frac{\partial^2\langle U\rangle}{\partial r^2}, \end{aligned}$$

which, using integration by parts, can be integrated to

$$0 = -\frac{1}{2}r^2\frac{\mathrm{d}p_w}{\mathrm{d}x} - \rho r\langle uv\rangle + \nu\rho\left(r\frac{\partial\langle U\rangle}{\partial r} - \langle U\rangle\right) + C,$$

where C is a constant of integration. At $r = 0$, symmetry implies that $\frac{\partial\langle U\rangle}{\partial r} = 0$, and hence $C = \nu\rho\langle U\rangle$, leading to

$$\tfrac{1}{2}r\frac{\mathrm{d}p_w}{\mathrm{d}x} = -\rho\langle uv\rangle + \nu\rho\frac{\partial\langle U\rangle}{\partial r} \equiv \tau(r).$$

At $r = R$, $\langle uv\rangle$ vanishes and thus

$$\frac{\mathrm{d}p_w}{\mathrm{d}x} = \frac{2}{R}\nu\rho\frac{\partial\langle U\rangle}{\partial r}\bigg|_{r=R} \equiv -2\frac{\tau_w}{R}.$$