

Turbulent Flows
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Cambridge University Press (2000)

Solution to Exercise 7.11

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Date: 12/17/17

In fully developed turbulent pipe flow, the mean temporal derivatives vanish and velocity statistics only depend on the radial coordinate. Hence, Eq. (5.45) reduces to

$$\frac{\partial}{\partial r}(r\langle V \rangle) = 0,$$

which, with the boundary condition $\langle V \rangle_{r=R} = 0$, integrates to

$$\langle V \rangle = 0.$$

The radial momentum equation (5.47) then simplifies to

$$\begin{aligned} 0 &= -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r}(r\langle v^2 \rangle) + \frac{\langle w^2 \rangle}{r} \\ &= -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial r} - \frac{\partial \langle v^2 \rangle}{\partial r} + \frac{\langle w^2 \rangle - \langle v^2 \rangle}{r}. \end{aligned}$$

Differentiating in the streamwise direction yields

$$0 = -\frac{1}{\rho} \frac{\partial^2 \langle p \rangle}{\partial x \partial r},$$

and thus

$$\frac{\partial \langle p \rangle}{\partial x} = \frac{dp_w}{dx} = \text{const.}$$

The axial momentum equation (5.46) simplifies to

$$\begin{aligned} 0 &= -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r}(r\langle uv \rangle) + \nu \frac{\partial^2 \langle U \rangle}{\partial r^2} \\ &= -r \frac{dp_w}{dx} - \rho \frac{\partial}{\partial r}(r\langle uv \rangle) + r\nu\rho \frac{\partial^2 \langle U \rangle}{\partial r^2}, \end{aligned}$$

which, using integration by parts, can be integrated to

$$0 = -\frac{1}{2}r^2 \frac{dp_w}{dx} - \rho r \langle uv \rangle + \nu \rho \left(r \frac{\partial \langle U \rangle}{\partial r} - \langle U \rangle \right) + C,$$

where C is a constant of integration. At $r = 0$, symmetry implies that $\frac{\partial \langle U \rangle}{\partial r} = 0$, and hence $C = \nu \rho \langle U \rangle$, leading to

$$\frac{1}{2}r \frac{dp_w}{dx} = -\rho \langle uv \rangle + \nu \rho \frac{\partial \langle U \rangle}{\partial r} \equiv \tau(r).$$

At $r = R$, $\langle uv \rangle$ vanishes and thus

$$\frac{dp_w}{dx} = \frac{2}{R} \nu \rho \frac{\partial \langle U \rangle}{\partial r} \Big|_{r=R} \equiv -2 \frac{\tau_w}{R}.$$

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