

**Turbulent Flows**  
 Stephen B. Pope  
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**Solution to Exercise 7.15**

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Since  $\text{Re} \stackrel{(7.95)}{=} \frac{D\bar{U}}{\nu}$ , we first have to determine the mean velocity. This can be done by assuming that the log law holds over the entire cross-section and integrating it by parts:

$$\begin{aligned}
 \frac{\bar{U}}{u_\tau} &\equiv \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R r \frac{\langle U \rangle}{u_\tau} dr d\theta \\
 &= \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R r \left( \frac{1}{\kappa} \ln \left( \frac{y u_\tau}{\nu} \right) + B \right) dr d\theta \\
 &= \frac{2}{R^2} \int_R^0 (y - R) \left( \frac{1}{\kappa} \ln \left( \frac{y u_\tau}{\nu} \right) \right) dy + B \\
 &= \frac{2}{R^2} \frac{1}{\kappa} \left( \left[ \left( \frac{1}{2} y^2 - R y \right) \ln \left( \frac{y u_\tau}{\nu} \right) \right]_{y=R}^{y=0} - \int_R^0 \left( \frac{1}{2} y - R \right) dy \right) + B \\
 &= \frac{2}{R^2} \frac{1}{\kappa} \left[ \left( \frac{1}{2} y^2 - R y \right) \ln \left( \frac{y u_\tau}{\nu} \right) - \frac{1}{4} y^2 + R y \right]_{y=R}^{y=0} + B \\
 &= \frac{1}{\kappa} \left( \ln \left( \frac{R u_\tau}{\nu} \right) - \frac{3}{2} \right) + B.
 \end{aligned}$$

From Eq. (7.104), we know that

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{8}} \frac{\bar{U}}{u_\tau}, \quad u_\tau = \bar{U} \sqrt{\frac{f}{8}},$$

and thus

$$\begin{aligned}
\frac{1}{\sqrt{f}} &\stackrel{(7.104)}{=} \frac{1}{\sqrt{8\kappa}} \left( \ln \left( \frac{R\bar{U}\sqrt{f}}{\sqrt{8\nu}} \right) - \frac{3}{2} \right) + \frac{B}{\sqrt{8}} \\
&= \frac{1}{\sqrt{8\kappa}} \ln \left( \frac{1}{\sqrt{32}} \operatorname{Re}\sqrt{f} \right) - \frac{3 - 2\kappa B}{2\sqrt{8\kappa}} \\
&= \frac{1}{2\sqrt{2\kappa}} \left( \ln \left( \operatorname{Re}\sqrt{f} \right) - \frac{1}{2} \ln(2^5) \right) - \frac{3 - 2\kappa B}{4\sqrt{2\kappa}} \\
&= \frac{1}{2\sqrt{2\kappa}} \left( \ln \left( \operatorname{Re}\sqrt{f} \right) \right) - \frac{3 + 5 \ln(2) - 2\kappa B}{4\sqrt{2\kappa}}.
\end{aligned}$$

Evaluating the coefficients for  $\kappa = 0.41$ ,  $B = 5.2$  and changing the base of the logarithm yields

$$\frac{1}{\sqrt{f}} \approx 1.99 \log_{10} \left( \operatorname{Re}\sqrt{f} \right) - 0.95.$$

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