Turbulent Flows
Stephen B. Pope
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Solution to Exercise 7.15

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Since Re $\overset{(7.95)}{=} \frac{D\bar{U}}{\nu}$, we first have to determine the mean velocity. This can be done by assuming that the log law holds over the entire cross-section and integrating it by parts:

$$\begin{aligned}
\frac{\bar{U}}{u_\tau} &\equiv \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R r\frac{\langle U\rangle}{u_\tau}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R r\left(\frac{1}{\kappa}\ln\left(\frac{yu_\tau}{\nu}\right)+B\right)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{2}{R^2}\int_R^0 (y-R)\left(\frac{1}{\kappa}\ln\left(\frac{yu_\tau}{\nu}\right)\right)\,\mathrm{d}y + B \\
&= \frac{2}{R^2}\frac{1}{\kappa}\left(\left[(\tfrac{1}{2}y^2-Ry)\ln\left(\frac{yu_\tau}{\nu}\right)\right]_{y=R}^{y=0} - \int_R^0(\tfrac{1}{2}y-R)\,\mathrm{d}y\right)+B \\
&= \frac{2}{R^2}\frac{1}{\kappa}\left[(\tfrac{1}{2}y^2-Ry)\ln\left(\frac{yu_\tau}{\nu}\right)-\frac{1}{4}y^2+Ry\right]_{y=R}^{y=0}+B \\
&= \frac{1}{\kappa}\left(\ln\left(\frac{Ru_\tau}{\nu}\right)-\frac{3}{2}\right)+B.
\end{aligned}$$

From Eq. (7.104), we know that

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{8}}\frac{\bar{U}}{u_\tau}, \quad u_\tau = \bar{U}\sqrt{\frac{f}{8}},$$

and thus

$$
\begin{aligned}
\frac{1}{\sqrt{f}} &\overset{(7.104)}{=} \frac{1}{\sqrt{8}\kappa}\left(\ln\left(\frac{R\bar{U}\sqrt{f}}{\sqrt{8}\nu}\right) - \frac{3}{2}\right) + \frac{B}{\sqrt{8}} \\
&= \frac{1}{\sqrt{8}\kappa}\ln\left(\frac{1}{\sqrt{32}}\mathrm{Re}\sqrt{f}\right) - \frac{3-2\kappa B}{2\sqrt{8}\kappa} \\
&= \frac{1}{2\sqrt{2}\kappa}\left(\ln\left(\mathrm{Re}\sqrt{f}\right) - \tfrac{1}{2}\ln(2^5)\right) - \frac{3-2\kappa B}{4\sqrt{2}\kappa} \\
&= \frac{1}{2\sqrt{2}\kappa}\left(\ln\left(\mathrm{Re}\sqrt{f}\right)\right) - \frac{3+5\ln(2)-2\kappa B}{4\sqrt{2}\kappa}.
\end{aligned}
$$

Evaluating the coefficients for $\kappa = 0.41, B = 5.2$ and changing the base of the logarithm yields

$$\frac{1}{\sqrt{f}} \approx 1.99\log_{10}\left(\mathrm{Re}\sqrt{f}\right) - 0.95.$$