Turbulent Flows
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Solution to Exercise 7.17

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The scaled bulk velocity $\frac{\bar{U}}{u_\tau}$ can be obtained for the log law in the fully rough regime from Eq. (7.119):

$$\frac{\bar{U}}{u_\tau} = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} r \left(\frac{1}{\kappa} \ln\left(\frac{y}{s}\right) + B_2 \right) \mathrm{d}r \, \mathrm{d}\theta.$$

Substituting $r = R - y, \mathrm{d}r = -\mathrm{d}y$ and subsequent integration by parts yields

$$\begin{aligned}
\frac{\bar{U}}{u_\tau} &= \frac{1}{\kappa} \frac{2}{R^2} \int_R^0 (y - R) \ln\left(\frac{y}{s}\right) \mathrm{d}y \, + B_2 \\
&= \frac{1}{\kappa} \frac{2}{R^2} \int_R^0 (y - R) \ln\left(\frac{y}{R}\right) \mathrm{d}y, \\
&= -\frac{1}{\kappa} \frac{2}{R^2} \left(\left[\ln\left(\frac{y}{s}\right) \left(\tfrac{1}{2} y^2 - Ry\right) \right]_{y=R}^{y=0} - \int_R^0 \left(\tfrac{1}{2} y - R\right) \mathrm{d}y \right) + B_2 \\
&= \frac{1}{\kappa} \frac{2}{R^2} \left[\ln\left(\frac{y}{s}\right) \left(\tfrac{1}{2} y^2 - Ry\right) - \tfrac{1}{4} y^2 + Ry \right]_{y=R}^{y=0} + B_2 \\
&= \frac{1}{\kappa} \ln\left(\frac{R}{s}\right) + B_2 - \frac{3}{2\kappa}.
\end{aligned}$$

From Eq. (7.104), we know that

$$\begin{aligned}
f &= 8 \left(\frac{\bar{U}}{u_\tau}\right)^{-2} \\
&= 8 \left[\frac{1}{\kappa} \ln\left(\frac{R}{s}\right) + B_2 - \frac{3}{2\kappa} \right]^{-2},
\end{aligned}$$

which can be reexpressed using $\kappa = 0.41, B_2 = 8.5$ as

$$
\begin{aligned}
f &= \left(\frac{\sqrt{8}}{\frac{1}{\kappa}\ln(10)\log_{10}\left(\frac{R}{s}\right) + B_2 - \frac{3}{2\kappa}}\right)^2 \\
&\approx \frac{1}{\left[1.99\log_{10}\left(\frac{R}{s}\right) + 1.71\right]^2}.
\end{aligned}
$$

Evaluating the asymptotic friction factors for the values of R/s given in Fig. 7.23 yields very good agreement with the experimental data, as can be seen from the following table.

Table 1: Asymptotic values of the friction factor, computed from Eq. (7.124) for the values of R/s given in Fig. 7.23.

R/s	15	31	60	126	252	507
f	0.061	0.046	0.036	0.0288	0.024	0.020