Turbulent Flows
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Solution to Exercise 7.19

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According to the mixing length hypothesis, as shown in Equation (7.87) of text, the eddy viscosity is,

$$\nu_T = \ell_m^2 \left| \frac{d\langle U\rangle}{dy} \right|, \tag{1}$$

where ℓ_m is the mixing length, $\langle U\rangle$ the time-averaged streamwise velocity, and y the wall-normal coordinate. It follows then that the Reynolds stress, $-\langle uv\rangle$, is approximated as,

$$-\langle uv\rangle = \nu_T \frac{d\langle U\rangle}{dy} = \ell_m^2 \left| \frac{d\langle U\rangle}{dy} \right| \frac{d\langle U\rangle}{dy}. \tag{2}$$

The expression $d\langle U\rangle/dy$ can be rewritten as,

$$\begin{aligned} \frac{d\langle U\rangle}{dy} &= \frac{d\langle U\rangle}{dy} \frac{u_\tau}{u_\tau} \frac{\delta_\nu}{\delta_\nu} \\ &= \frac{d(\langle U\rangle/u_\tau)}{d(y/\delta_\nu)} \frac{u_\tau}{\delta_\nu} \\ &= \frac{du^+}{dy^+} \frac{u_\tau}{\delta_\nu} \\ &= 1 \cdot \frac{u_\tau}{\delta_\nu} = \frac{u_\tau^2}{\nu}. \end{aligned} \tag{3}$$

In the foregoing equation, u_τ is the friction velocity and δ_ν is the viscous lengthscale. Also, $du^+/dy^+ = 1$ for $y^+ \ll 1$ in the viscous sublayer; see Equation (7.40) of text. With the expression for $d\langle U\rangle/dy$ in Equation (3), Equation (2) normalized by the square of u_τ can be reexpressed as,

$$\begin{aligned} \frac{-\langle uv\rangle}{u_\tau^2} &= \ell_m^2 \left| \frac{u_\tau^2}{\nu} \right| \frac{u_\tau^2}{\nu} \frac{1}{u_\tau^2} \\ &= \ell_m^2 \frac{u_\tau^2}{\nu^2} = \frac{\ell_m^2}{\delta_\nu^2} = (\ell_m^+)^2. \end{aligned} \tag{4}$$

In Equation (4), $\ell_m^+ = \ell_m/\delta_\nu$ is the mix-length in viscous scales. And the shear stress would be positive with the assumption of attached flow.

The second part of the question is based on the van Driest approximation, where

$$\ell_m^+ = \kappa y^+[1 - \exp(-y^+/A^+)]. \tag{5}$$

Using the result from Equation (4),

$$(\ell_m^+)^2 = \kappa^2 (y^+)^2 [1 - 2\exp(-y^+/A^+) + \exp(-2y^+/A^+)]. \tag{6}$$

Recall the Taylor series expansion for e^x as,

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5). \tag{7}$$

Therefore, Equation (6) can be rewritten as,

$$\begin{aligned}(\ell_m^+)^2 = \kappa^2 (y^+)^2 \Bigg\{ 1 - 2\Bigg[1 - \frac{y^+}{A^+} + \frac{1}{2!}\left(\frac{y^+}{A^+}\right)^2 + \mathcal{O}\left(\left(\frac{y^+}{A^+}\right)^3\right) \\ + \left(1 - 2\frac{y^+}{A^+} + \frac{1}{2!}\left(\frac{2y^+}{A^+}\right)^2 + \mathcal{O}\left(\left(\frac{y^+}{A^+}\right)^3\right)\right)\Bigg]\Bigg\}. \end{aligned} \tag{8}$$

Simplifying the above expression,

$$(\ell_m^+)^2 = \kappa^2 (y^+)^2 \left[-\left(\frac{y^+}{A^+}\right)^2 + 2\left(\frac{y^+}{A^+}\right)^2 + \mathcal{O}\left(\left(\frac{y^+}{A^+}\right)^3\right)\right]. \tag{9}$$

Neglecting higher order terms, the desired form of the solution is,

$$(\ell_m^+)^2 = \kappa^2 (y^+)^2 \left(\frac{y^+}{A^+}\right)^2. \tag{10}$$

Finally,

$$\frac{-\langle uv \rangle}{u_\tau^2} = (\ell_m^+)^2 = \frac{\kappa^2}{(A^+)^2}(y^+)^4. \tag{11}$$

The correct expression for the near-wall Reynolds shear stress is, according to Equation (7.63),

$$\langle uv \rangle = \langle b_1 c_3 \rangle y^3 + \mathcal{O}(y^4),$$

where b_1 and c_3 are constants. Evidently, the form of the near-wall Reynolds stresses deduced based on boundary conditions suggests an asymptotic behavior similar to a third order polynomial, which differs from the polynomial derived based on the van Driest approximation. It seems that the van Driest function for mixing length is not exactly correct for $y^+ \ll 1$.