Turbulent Flows
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Cambridge University Press (2000)

Solution to Exercise 7.26

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a:) The trace of $\mathcal{P}_{ij}$ is

$$\begin{aligned}
\mathcal{P}_{ii} &= \left(-\langle u_i u_k\rangle \frac{\partial \langle U_j\rangle}{\partial x_k} - \langle u_j u_k\rangle \frac{\partial \langle U_i\rangle}{\partial x_k}\right)\delta_{ij} \\
&= -\langle u_j u_k\rangle \frac{\partial \langle U_j\rangle}{\partial x_k} - \langle u_j u_k\rangle \frac{\partial \langle U_j\rangle}{\partial x_k} \\
&= -2\langle u_j u_k\rangle \frac{\partial \langle U_j\rangle}{\partial x_k} \\
&= 2\mathcal{P}, \qquad (1)
\end{aligned}$$

i.e.,

$$\frac{1}{2}\mathcal{P}_{ii} = \mathcal{P}. \qquad (2)$$

The trace of ε_{ij} is

$$\begin{aligned}
\varepsilon_{ii} &= 2\nu \langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\rangle \delta_{ij} \\
&= 2\nu \langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}\rangle \\
&= 2\tilde{\varepsilon}, \qquad (3)
\end{aligned}$$

i.e.,

$$\frac{1}{2}\varepsilon_{ii} = \tilde{\varepsilon}. \qquad (4)$$

b:) According to Eq.(7.194), the Reynolds-stress equation can be written

$$\frac{\bar{D}\langle u_i u_j\rangle}{\bar{D}t} + \frac{\partial}{\partial x_k} T_{kij} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \varepsilon_{ij}, \qquad (5)$$

where the Reynolds-stress flux T_{kij} is

$$T_{kij} = T_{kij}^{(u)} + T_{kij}^{(p)} + T_{kij}^{(\nu)}, \tag{6}$$

with

$$T_{kij}^{(u)} \equiv \langle u_i u_j u_k \rangle, \qquad T_{kij}^{(\nu)} = -\nu \frac{\partial \langle u_i u_j \rangle}{\partial x_k}. \tag{7}$$

Multiplying Eq. 5 with δ_{ij}, we obtain

$$\delta_{ij} \left(\frac{\bar{D} \langle u_i u_j \rangle}{\bar{D} t} + \frac{\partial}{\partial x_k} T_{kij} \right) = \delta_{ij} \left(\mathcal{P}_{ij} + \mathcal{R}_{ij} - \varepsilon_{ij} \right), \tag{8}$$

i.e.,

$$\frac{\bar{D}}{\bar{D} t} (2k) + \frac{\partial}{\partial x_k} \left(\langle u_k u_j u_j \rangle - \nu \frac{\partial 2k}{\partial x_k} + \frac{2}{\rho} \langle u_k p' \rangle \right) = 2\mathcal{P} - 2\tilde{\varepsilon}, \tag{9}$$

or, dividing by 2,

$$\frac{\bar{D} k}{\bar{D} t} + \frac{\partial}{\partial x_k} \left(\frac{1}{2} \langle u_k u_j u_j \rangle + \frac{\langle u_k p' \rangle}{\rho} \right) = \nu \nabla^2 k + \mathcal{P} - \tilde{\varepsilon}, \tag{10}$$

So half the trace of the Reynolds-stress equation (Eq.(7.194)) is identical to the kinetic energy equation (Eq.(5.164)).