

Turbulent Flows
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Cambridge University Press (2000)

Solution to Exercise 7.26

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Date: 04/17/03

a:) The trace of \mathcal{P}_{ij} is

$$\begin{aligned}\mathcal{P}_{ii} &= \left(-\langle u_i u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} - \langle u_j u_k \rangle \frac{\partial \langle U_i \rangle}{\partial x_k} \right) \delta_{ij} \\ &= -\langle u_j u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} - \langle u_j u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} \\ &= -2\langle u_j u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} \\ &= 2\mathcal{P},\end{aligned}\tag{1}$$

i.e.,

$$\frac{1}{2}\mathcal{P}_{ii} = \mathcal{P}.\tag{2}$$

The trace of ε_{ij} is

$$\begin{aligned}\varepsilon_{ii} &= 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle \delta_{ij} \\ &= 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right\rangle \\ &= 2\tilde{\varepsilon},\end{aligned}\tag{3}$$

i.e.,

$$\frac{1}{2}\varepsilon_{ii} = \tilde{\varepsilon}.\tag{4}$$

b:) According to Eq.(7.194), the Reynolds-stress equation can be written

$$\frac{\bar{D}\langle u_i u_j \rangle}{\bar{D}t} + \frac{\partial}{\partial x_k} T_{kij} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \varepsilon_{ij},\tag{5}$$

where the Reynolds-stress flux T_{kij} is

$$T_{kij} = T_{kij}^{(u)} + T_{kij}^{(p)} + T_{kij}^{(\nu)}, \quad (6)$$

with

$$T_{kij}^{(u)} \equiv \langle u_i u_j u_k \rangle, \quad T_{kij}^{(\nu)} = -\nu \frac{\partial \langle u_i u_j \rangle}{\partial x_k}. \quad (7)$$

Multiplying Eq. 5 with δ_{ij} , we obtain

$$\delta_{ij} \left(\frac{\bar{D} \langle u_i u_j \rangle}{\bar{D}t} + \frac{\partial}{\partial x_k} T_{kij} \right) = \delta_{ij} (\mathcal{P}_{ij} + \mathcal{R}_{ij} - \varepsilon_{ij}), \quad (8)$$

i.e.,

$$\frac{\bar{D}}{\bar{D}t} (2k) + \frac{\partial}{\partial x_k} \left(\langle u_k u_j u_j \rangle - \nu \frac{\partial 2k}{\partial x_k} + \frac{2}{\rho} \langle u_k p' \rangle \right) = 2\mathcal{P} - 2\tilde{\varepsilon}, \quad (9)$$

or, dividing by 2,

$$\frac{\bar{D}k}{\bar{D}t} + \frac{\partial}{\partial x_k} \left(\frac{1}{2} \langle u_k u_j u_j \rangle + \frac{\langle u_k p' \rangle}{\rho} \right) = \nu \nabla^2 k + \mathcal{P} - \tilde{\varepsilon}, \quad (10)$$

So half the trace of the Reynolds-stress equation (Eq.(7.194)) is identical to the kinetic energy equation (Eq.(5.164)).

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