Turbulent Flows
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Solution to Exercise 7.28

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a:) With Reynolds-stress anisotropy defined by $a_{ij} \equiv \langle u_i u_j \rangle - 2/3 k \delta_{ij}$, we obtain

$$\begin{aligned}
\mathcal{P}_{ij} &= -\langle u_i u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} - \langle u_j u_k \rangle \frac{\partial \langle U_i \rangle}{\partial x_k} \\
&= -\langle u_i u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} - \langle u_j u_k \rangle \frac{\partial \langle U_i \rangle}{\partial x_k} \\
&\quad -\frac{2}{3} k \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) + \frac{2}{3} k \delta_{ik} \frac{\partial \langle U_j \rangle}{\partial x_k} + \frac{2}{3} k \delta_{jk} \frac{\partial \langle U_i \rangle}{\partial x_k} \\
&= -\frac{2}{3} k \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - \langle u_i u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} + \frac{2}{3} k \delta_{ik} \frac{\partial \langle U_j \rangle}{\partial x_k} \\
&\quad -\langle u_j u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} + \frac{2}{3} k \delta_{jk} \frac{\partial \langle U_i \rangle}{\partial x_k} \\
&= -\frac{2}{3} k \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - \left(\langle u_i u_k \rangle - \frac{2}{3} k \delta_{ik} \right) \frac{\partial \langle U_j \rangle}{\partial x_k} \\
&\quad - \left(\langle u_j u_k \rangle - \frac{2}{3} k \delta_{jk} \right) \frac{\partial \langle U_i \rangle}{\partial x_k} \\
&= -\frac{2}{3} k \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - a_{ik} \frac{\partial \langle U_j \rangle}{\partial x_k} - a_{jk} \frac{\partial \langle U_i \rangle}{\partial x_k}. \qquad (1)
\end{aligned}$$

So in terms of the Reynolds-stress anisotropy, the production tensor is

$$\mathcal{P}_{ij} = -\frac{2}{3} k \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - a_{ik} \frac{\partial \langle U_j \rangle}{\partial x_k} - a_{jk} \frac{\partial \langle U_i \rangle}{\partial x_k}. \qquad (2)$$

b:) For initially isotropic turbulence, if it is subject to axisymmetric contraction (see section 10.1.1), then in the axisymmetric contraction section, $\mathcal{P}_{11}$ is negative, i.e.,

$$\mathcal{P}_{11} = -\frac{4}{3}k\frac{\partial\langle U_1\rangle}{\partial x_1} < 0, \quad for \quad \frac{\partial\langle U_1\rangle}{\partial x_1} > 0. \tag{3}$$