Turbulent Flows
Stephen B. Pope
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Solution to Exercise 7.30

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The definition of ε_{ij} is

$$\varepsilon_{ij} \equiv 2\nu\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\rangle. \tag{1}$$

For very small y (very close to the wall), the fluctuating velocity components can be written as Taylor series of the forms (see section 7.1.6)

$$u = b_1 y + c_1 y^2 + ..., \tag{2}$$

$$v = c_2 y^2 + ..., \tag{3}$$

$$w = b_3 y + c_3 y^2. \tag{4}$$

Now, at the wall (y=0), all components of $\mathbf{u}$ are zero, as are the derivatives $\partial u_i/\partial x_1$ and $\partial u_i/\partial x_3$. The only non-zero derivatives are $\partial u_i/\partial x_2 = \partial u_i/\partial y$. Thus we obtain

$$\begin{aligned}
\varepsilon_{ij} &= 2\nu\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\rangle = 2\nu\langle \frac{\partial u_i}{\partial y}\frac{\partial u_j}{\partial y}\rangle \\
&= \nu\frac{\partial^2\langle u_i u_j\rangle}{\partial y^2} - \langle u_i\frac{\partial^2 u_j}{\partial y^2}\rangle - \langle u_j\frac{\partial^2 u_i}{\partial y^2}\rangle \\
&= \nu\frac{\partial^2\langle u_i u_j\rangle}{\partial y^2}.
\end{aligned} \tag{5}$$

$$\begin{aligned}
\varepsilon_{11} &= 2\nu\langle \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\rangle = 2\nu\langle b_1^2\rangle, \\
\varepsilon_{22} &= 2\nu\langle \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\rangle = 0,
\end{aligned}$$

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\begin{aligned}
\varepsilon_{33} &= 2\nu\langle\frac{\partial w}{\partial y}\frac{\partial w}{\partial y}\rangle = 2\nu\langle b_3^2\rangle, \\
\varepsilon_{12} = \varepsilon_{21} &= 2\nu\langle\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\rangle = 0, \\
\varepsilon_{13} = \varepsilon_{31} &= 2\nu\langle\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\rangle = 2\nu\langle b_1 b_3\rangle, \\
\varepsilon_{23} = \varepsilon_{32} &= 2\nu\langle\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\rangle = 0. \qquad (6)
\end{aligned}
$$

In short, ε_{11}, ε_{33}, ε_{13} and ε_{31} are non-zero at the wall, and other components are zero at the wall.