Turbulent Flows
Stephen B. Pope
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Solution to Exercise 7.9

Prepared by: Shi Jin *Date:* 3/10/03

a) There is Taylor's series expansion for $\langle U \rangle$ at the wall:

$$\begin{aligned} \langle U \rangle &= \langle U \rangle_{y=0} + \left(\frac{d \langle U \rangle}{dy} \right)_{y=0} y + \left(\frac{1}{2} \frac{d^2 \langle U \rangle}{dy^2} \right)_{y=0} y^2 \\ &+ \left(\frac{1}{6} \frac{d^3 \langle U \rangle}{dy^3} \right)_{y=0} y^3 + \left(\frac{1}{24} \frac{d^4 \langle U \rangle}{dy^4} \right)_{y=0} y^4 + ... \end{aligned} \tag{1}$$

We know from boundary condition that

$$\langle U \rangle_{y=0} = 0, \tag{2}$$

and from Eq.(7.24) that

$$\left(\frac{d \langle U \rangle}{dy} \right)_{y=0} = \frac{\tau_w}{\rho \nu}. \tag{3}$$

According to Eqs.(7.10) and (7.13), we have

$$\begin{aligned} \tau(y) &= \rho \nu \frac{d \langle U \rangle}{dy} - \rho \langle uv \rangle \\ &= \tau_w \left(1 - \frac{y}{\delta} \right), \end{aligned} \tag{4}$$

hence

$$\begin{aligned} \frac{d \langle U \rangle}{dy} &= \frac{\tau(y) + \rho \langle uv \rangle}{\rho \nu} \\ &= \frac{\tau_w}{\rho \nu} - \frac{\tau_w y}{\rho \nu \delta} + \frac{\langle uv \rangle}{\nu}. \end{aligned} \tag{5}$$

Using the relationship of Eq.(7.76)

$$\langle uv \rangle = -\sigma {u_\tau}^2 {y^+}^3 ... \tag{6}$$

and the definition of y^+ and u_τ, we obtain

$$\frac{d^2 \langle U \rangle}{dy^2} = -\frac{{u_\tau}^2}{\nu\delta} - \frac{3\sigma {u_\tau}^3}{\nu} {y^+}^2, \tag{7}$$

$$\frac{d^3 \langle U \rangle}{dy^3} = -\frac{6\sigma {u_\tau}^4}{\nu^3} y^+, \tag{8}$$

$$\frac{d^4 \langle U \rangle}{dy^4} = -\frac{6\sigma {u_\tau}^5}{\nu^4}. \tag{9}$$

Put them back into Eq(1) and using the definition of u^+ and Re_τ, we obtain

$$\begin{aligned} u^+ &= \frac{\langle U \rangle}{u_\tau} \\ &= y^+ - \frac{{y^+}^2}{2Re_\tau} - \frac{1}{4}\sigma {y^+}^4 .... \end{aligned} \tag{10}$$

b) When Re_τ is large enough, the law of the wall becomes

$$\begin{aligned} u^+ &= f_w\left(y^+\right) \\ &= y^+ - \frac{1}{4}\sigma {y^+}^4 .... \end{aligned} \tag{11}$$