

**Turbulent Flows**  
Stephen B. Pope  
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**Solution to Exercise 7.9**

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a) There is Taylor's series expansion for  $\langle U \rangle$  at the wall:

$$\begin{aligned} \langle U \rangle &= \langle U \rangle_{y=0} + \left( \frac{d\langle U \rangle}{dy} \right)_{y=0} y + \left( \frac{1}{2} \frac{d^2\langle U \rangle}{dy^2} \right)_{y=0} y^2 \\ &+ \left( \frac{1}{6} \frac{d^3\langle U \rangle}{dy^3} \right)_{y=0} y^3 + \left( \frac{1}{24} \frac{d^4\langle U \rangle}{dy^4} \right)_{y=0} y^4 + \dots \end{aligned} \quad (1)$$

We know from boundary condition that

$$\langle U \rangle_{y=0} = 0, \quad (2)$$

and from Eq.(7.24) that

$$\left( \frac{d\langle U \rangle}{dy} \right)_{y=0} = \frac{\tau_w}{\rho\nu}. \quad (3)$$

According to Eqs.(7.10) and (7.13), we have

$$\begin{aligned} \tau(y) &= \rho\nu \frac{d\langle U \rangle}{dy} - \rho \langle uv \rangle \\ &= \tau_w \left( 1 - \frac{y}{\delta} \right), \end{aligned} \quad (4)$$

hence

$$\begin{aligned} \frac{d\langle U \rangle}{dy} &= \frac{\tau(y) + \rho \langle uv \rangle}{\rho\nu} \\ &= \frac{\tau_w}{\rho\nu} - \frac{\tau_w y}{\rho\nu\delta} + \frac{\langle uv \rangle}{\nu}. \end{aligned} \quad (5)$$

Using the relationship of Eq.(7.76)

$$\langle uv \rangle = -\sigma u_\tau^2 y^{+3} \dots \quad (6)$$

and the definition of  $y^+$  and  $u_\tau$ , we obtain

$$\frac{d^2 \langle U \rangle}{dy^2} = -\frac{u_\tau^2}{\nu \delta} - \frac{3\sigma u_\tau^3}{\nu} y^{+2}, \quad (7)$$

$$\frac{d^3 \langle U \rangle}{dy^3} = -\frac{6\sigma u_\tau^4}{\nu^3} y^+, \quad (8)$$

$$\frac{d^4 \langle U \rangle}{dy^4} = -\frac{6\sigma u_\tau^5}{\nu^4}. \quad (9)$$

Put them back into Eq(1) and using the definition of  $u^+$  and  $Re_\tau$ , we obtain

$$\begin{aligned} u^+ &= \frac{\langle U \rangle}{u_\tau} \\ &= y^+ - \frac{y^{+2}}{2Re_\tau} - \frac{1}{4}\sigma y^{+4} \dots \end{aligned} \quad (10)$$

b) When  $Re_\tau$  is large enough, the law of the wall becomes

$$\begin{aligned} u^+ &= f_w(y^+) \\ &= y^+ - \frac{1}{4}\sigma y^{+4} \dots \end{aligned} \quad (11)$$

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