Turbulent Flows
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Solution to Exercise 5.40

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a) The Reynolds equation (Eq.(4.12)) is

$$\frac{\bar{\mathrm{D}}\langle U_j\rangle}{\bar{\mathrm{D}}t} = \nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}, \tag{1}$$

and the Navier-Stokes equation is

$$\frac{\mathrm{D}U_j}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_j x_j}. \tag{2}$$

By subtracting the Reynolds equation from the Navier-Stokes equation, we get that the fluctuating velocity $\mathbf{u}(\mathbf{x}, t)$ evolves by

$$\frac{DU_j}{Dt} - \frac{\bar{D}\langle U_j\rangle}{Dt} = \frac{\partial}{\partial x_i}\langle u_i u_j\rangle + \nu\nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}, \tag{3}$$

i.e.,

$$\frac{DU_j}{Dt} - \frac{D\langle U_j\rangle}{Dt} + u_i\frac{\partial\langle U_j\rangle}{\partial x_i} = \frac{\partial}{\partial x_i}\langle u_i u_j\rangle + \nu\nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}. \tag{4}$$

So from Eq. 4, we get

$$\frac{Du_j}{Dt} = -u_i\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial}{\partial x_i}\langle u_i u_j\rangle + \nu\nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}. \tag{5}$$

b) For homogeneous turbulence with $\nabla.\mathbf{U} = 0$,

$$\begin{aligned}\left\langle u_j\frac{Du_j}{Dt}\right\rangle &= \left\langle u_j\frac{\partial u_j}{\partial t} + u_j\langle U_i\rangle\frac{\partial u_j}{\partial x_i}\right\rangle \\ &= \left\langle \frac{1}{2}\frac{\partial(u_j u_j)}{\partial t} + \frac{1}{2}\frac{\partial(\langle U_i\rangle u_j u_j)}{\partial x_i}\right\rangle \\ &= \frac{\mathrm{d}k}{\mathrm{d}t}, \end{aligned} \tag{6}$$

and

$$\left\langle u_j \frac{\partial p'}{\partial x_j} \right\rangle = \frac{\partial \langle u_j p' \rangle}{\partial x_j} = 0, \tag{7}$$

and

$$\begin{aligned} \nu \langle u_j \nabla^2 u_j \rangle &= \nu \frac{\partial^2 \langle \frac{1}{2} u_j u_j \rangle}{\partial x_i \partial x_i} - \nu \left\langle \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} \right\rangle \\ &= -\varepsilon. \end{aligned} \tag{8}$$

Multiplying Eq. 5 by u_j and taking the mean, we obtain

$$\left\langle u_j \frac{D u_j}{D t} \right\rangle = -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} + \langle u_j \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \nu \langle u_j \nabla^2 u_j \rangle - \frac{1}{\rho} \langle u_j \frac{\partial p'}{\partial x_j} \rangle. \tag{9}$$

Substituting Eqs. 6, 7 and 8 into Eq. 9, we get that the kinetic energy evolves by

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \mathcal{P} - \varepsilon, \tag{10}$$

where

$$\mathcal{P} \equiv -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i}. \tag{11}$$

[Note that this implies that a necessary condition for homogeneous turbulence is that the mean velocity gradients be uniform.]