

**Turbulent Flows**  
Stephen B. Pope  
*Cambridge University Press* (2000)  
**Solution to Exercise 5.40**

*Prepared by:* Zhuyin Ren

*Date:* 05/07/03

a) The Reynolds equation (Eq.(4.12)) is

$$\frac{\bar{D}\langle U_j \rangle}{Dt} = \nu \nabla^2 \langle U_j \rangle - \frac{\partial \langle u_i u_j \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j}, \quad (1)$$

and the Navier-Stokes equation is

$$\frac{DU_j}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j}. \quad (2)$$

By subtracting the Reynolds equation from the Navier-Stokes equation, we get that the fluctuating velocity  $\mathbf{u}(\mathbf{x}, t)$  evolves by

$$\frac{DU_j}{Dt} - \frac{\bar{D}\langle U_j \rangle}{Dt} = \frac{\partial}{\partial x_i} \langle u_i u_j \rangle + \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\partial p'}{\partial x_j}, \quad (3)$$

i.e.,

$$\frac{Du_j}{Dt} - \frac{D\langle U_j \rangle}{Dt} + u_i \frac{\partial \langle U_j \rangle}{\partial x_i} = \frac{\partial}{\partial x_i} \langle u_i u_j \rangle + \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\partial p'}{\partial x_j}. \quad (4)$$

So from Eq. 4, we get

$$\frac{Du_j}{Dt} = -u_i \frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\partial}{\partial x_i} \langle u_i u_j \rangle + \nu \nabla^2 u_j - \frac{1}{\rho} \frac{\partial p'}{\partial x_j}. \quad (5)$$

b) For homogeneous turbulence with  $\nabla \cdot \mathbf{U} = 0$ ,

$$\begin{aligned} \left\langle u_j \frac{Du_j}{Dt} \right\rangle &= \left\langle u_j \frac{\partial u_j}{\partial t} + u_j \langle U_i \rangle \frac{\partial u_j}{\partial x_i} \right\rangle \\ &= \left\langle \frac{1}{2} \frac{\partial (u_j u_j)}{\partial t} + \frac{1}{2} \frac{\partial (\langle U_i \rangle u_j u_j)}{\partial x_i} \right\rangle \\ &= \frac{dk}{dt}, \end{aligned} \quad (6)$$

and

$$\left\langle u_j \frac{\partial p'}{\partial x_j} \right\rangle = \frac{\partial \langle u_j p' \rangle}{\partial x_j} = 0, \quad (7)$$

and

$$\begin{aligned} \nu \langle u_j \nabla^2 u_j \rangle &= \nu \frac{\partial^2 \langle \frac{1}{2} u_j u_j \rangle}{\partial x_i \partial x_i} - \nu \left\langle \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} \right\rangle \\ &= -\varepsilon. \end{aligned} \quad (8)$$

Multiplying Eq. 5 by  $u_j$  and taking the mean, we obtain

$$\left\langle u_j \frac{Du_j}{Dt} \right\rangle = -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} + \langle u_j \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \nu \langle u_j \nabla^2 u_j \rangle - \frac{1}{\rho} \langle u_j \frac{\partial p'}{\partial x_j} \rangle. \quad (9)$$

Substituting Eqs. 6, 7 and 8 into Eq. 9, we get that the kinetic energy evolves by

$$\frac{dk}{dt} = \mathcal{P} - \varepsilon, \quad (10)$$

where

$$\mathcal{P} \equiv -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i}. \quad (11)$$

[Note that this implies that a necessary condition for homogeneous turbulence is that the mean velocity gradients be uniform.]

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/1.0> or send a letter to Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.